Clock arithmetic is very much like school arithmetic. In particular, the following rules hold:

1) $a+b = b+a$ (commutativity)

1') $ab = ba$

2) $(a+b)+c = a+(b+c)$

2') $(ab)c = a(bc)$ (associativity)

3) $a(b+c) = ab+ac$ (distributivity)

We'll now study an example of an arithmetic where (1') does not hold.

Matrix Arithmetic

A matrix is an array of numbers arranged neatly in rows and columns. Each row has the same length, and each column has the same length.

Example

$$\begin{pmatrix} 1 & 2 & 5 \\ -2 & 3 & 1 \end{pmatrix}$$ 2×3 matrix

$$\begin{pmatrix} \frac{1}{2} & -\sqrt{2} \\ 3 & 5 \end{pmatrix}$$ 2×2 matrix

$$\begin{pmatrix} 1 & 2 & 3 & 4 & -5 \end{pmatrix}$$ 1×5 matrix

also a "row vector"

$$\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$ 3×1 matrix

also a "column vector"

Matrix Addition

Two $m \times n$ matrices A, B are **added** by adding corresponding entries.

$$\underset{A}{\begin{pmatrix} 1 & 3 & -2 \\ 2 & 4 & -7 \end{pmatrix}} + \underset{B}{\begin{pmatrix} -2 & 3 & 1 \\ 3 & 0 & 0 \end{pmatrix}} = \underset{A+B}{\begin{pmatrix} -1 & 6 & -1 \\ 5 & 4 & -7 \end{pmatrix}}$$

$$\underset{A}{\begin{pmatrix} -2 & 3 \\ 4 & 1 \end{pmatrix}} + \underset{B}{\begin{pmatrix} 1 & -3 \\ 4 & 6 \end{pmatrix}} = \underset{A+B}{\begin{pmatrix} -1 & 0 \\ 8 & 7 \end{pmatrix}}$$

$$\underset{A}{\begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}} + \underset{B}{\begin{pmatrix} -3, & -2 & 1, & 6 \end{pmatrix}} = \underset{A+B}{\begin{pmatrix} -2, & 0 & 4 & 10 \end{pmatrix}}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{pmatrix}$$

can <u>not</u> be added.

Given a matrix A we write $-A$ to denote the matrix got from A by placing a "$-$" in front of each entry.

$$A = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix}$$

$$-A = \begin{pmatrix} -1 & -2 \\ -3 & 4 \end{pmatrix}$$

Note

$A + (-A) =$ matrix with all zero entries

$$\begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} + \begin{pmatrix} -1 & -2 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

A matrix whose entries are all zero is called a zero matrix, and often denoted by 0.

Thus, for any matrix A we have

$$A + (-A) = 0.$$

Multiplication of a row vector by a column vector.

Let

$$R = (a_1, a_2, \dots, a_n)$$

be a row vector of length n.

Let

$$C = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

be a column vector of length n.

We define

$$R.C = (a_1 \; a_2 \cdots a_n) \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$= a_1 b_1 + a_2 b_2 + a_3 b_3 + \cdots + a_n b_n.$$

<u>Example</u> $R = (-1, 2, -3)$, $C = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}$

$$\begin{aligned} R.C &= (-1)(2) + (2)(0) + (-3)(4) \\ &= -2 + 0 - 12 \\ &= -14. \end{aligned}$$

Matrix Multiplication

A matrix

$$A = \begin{array}{cc} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} & \begin{array}{c} R_1 \\ R_2 \end{array} \end{array}$$

can be regarded as a collection of rows R_i.

A matrix

$$B = \begin{array}{c} \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} \\ \begin{pmatrix} -1 & 2 & 1 \\ 1 & 3 & 3 \\ 0 & 2 & 4 \end{pmatrix} \end{array}$$

can be thought of as a collection of columns C_j

Let A be an $m \times \underline{n}$ matrix.
Let B be an $\underline{n} \times p$ matrix
We define the product

$$AB$$

as

$$\underbrace{\begin{pmatrix} — & R_1 & — \\ — & R_2 & — \\ & \vdots & \\ — & R_m & — \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} | & | & & | \\ c_1 & c_2 & \cdots & c_p \\ | & | & & | \end{pmatrix}}_{B}$$

$$= \underbrace{\begin{pmatrix} R_1C_1 & R_1C_2 & \cdots & R_1C_p \\ R_2C_1 & R_2C_2 & \cdots & R_2C_p \\ \vdots & & & \vdots \\ R_mC_1 & \cdots & & R_mC_p \end{pmatrix}}_{AB}$$

Example

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

$$B = \begin{pmatrix} -1 & 2 & 1 \\ 1 & 3 & 3 \\ 0 & 2 & 0 \end{pmatrix}$$

Calculate AB

Sol^n

$$\begin{pmatrix} 1 & 14 & 7 \\ 1 & 3.5 & 14 \end{pmatrix}$$

$$(1\ 2\ 3)\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = -1 + 2 + 0 = 1$$

$$(1\ 2\ 3)\begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} = 2 + 6 + 6 = 14$$

$$(4\ 5\ 6)\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = -4 + 5 + 0 = 1$$