Why does the Gauss-Jordan method succeed in finding the inverse A^{-1} of a square matrix A.

To answer this we need to understand row operations.

Row operation I

e.g.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 8 & 6 \end{pmatrix} \xrightarrow{R_2 \mapsto R_2 - 2R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 3 & 8 & 6 \end{pmatrix} = B$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{E} \underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 8 & 6 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 3 & 8 & 6 \end{pmatrix}}_{B}$$

Row Operation II

e.g.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 8 & 6 \end{pmatrix} \xleftrightarrow{R_2 \Leftrightarrow R_3} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 8 & 6 \\ 2 & 5 & 5 \end{pmatrix}$$

$$A \qquad\qquad\qquad\qquad B$$

Also

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}_{E} \underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 8 & 6 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 3 & 8 & 6 \\ 2 & 5 & 5 \end{pmatrix}}_{B}$$

Row Operation III

e.g.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 8 & 6 \end{pmatrix} \xrightarrow{R_3 \mapsto -2R_3} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ -6 & -16 & -12 \end{pmatrix}$$

$$A \qquad\qquad\qquad\qquad B$$

Also

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}}_{E} \underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 8 & 6 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ -6 & -16 & -12 \end{pmatrix}}_{B}$$

Let's look again at the Gauss-Jordan method for finding A^{-1}.

If

$$(A \mid I) \xrightarrow[\text{operations}]{\text{row}} (I \mid B)$$

then there are matrices $E_1, E_2, \ldots, E_k$ such that

$$(E_k \cdots E_3 E_2 E_1) A = I$$

so

$$(E_k \cdots E_2 E_1) A A^{-1} = I A^{-1}$$

and

$$\underbrace{(E_k \cdots E_2 E_1) I}_{B} = A^{-1} \quad (*).$$

Note: $(*)$ says $B = A^{-1}$.

This (kind of) proves why the Gauss-Jordan method works.

Example A factory requires energy, steel and labour to manufacture machines of type A, B and C.

Resource	A	B	C	weekly weekly
energy	2 Mwh	3 Mwh	2 Mwh	100 Mwh
steel	1 tonne	1 tonne	4 tonne	70 tonnes
labour	20 hrs	10 hrs	10 hrs	500 hrs

What production figures ensure that all resources are used.

Sol^n Let's suppose we manufacture

x units of machine A

y " " " B

z " " " C

If all resources are used then

$$\left.\begin{aligned} 2x + 3y + 2z &= 100 \\ x + y + 4z &= 70 \\ 20x + 10y + 10z &= 500 \end{aligned}\right\} (*)$$

Sol^n

$$\begin{pmatrix} 2 & 3 & 2 \\ 1 & 1 & 4 \\ 20 & 10 & 10 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 100 \\ 70 \\ 500 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 & 2 \\ 1 & 1 & 4 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 100 \\ 70 \\ 50 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 4 \\ 2 & 3 & 2 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 70 \\ 100 \\ 50 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 4 \\ 0 & 1 & -6 \\ 0 & -1 & -7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 70 \\ -40 \\ -90 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 4 \\ 0 & 1 & -6 \\ 0 & 0 & -13 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 70 \\ -40 \\ -130 \end{pmatrix}$$

$$\begin{aligned} x + y + 4z &= 70 \\ y - 6z &= -14 \\ -13z &= -130 \end{aligned}$$

$$\boxed{\begin{aligned} z &= 10 \\ y &= 20 \\ x &= 10 \end{aligned}}$$

$$\begin{aligned} y &= -\cancel{14}40 + 6z \\ &= -\cancel{14}40 + 60 \\ &= 20 \end{aligned}$$

$$\begin{aligned} x &= 70 - y - 4z \\ &= 70 - 20 - 40 \\ &= 10 \end{aligned}$$