Last week: we checked that

$T: \mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \longmapsto (x+2y, 3x+4y)$

was linear. i.e.

$$T(P+Q) = T(P) + T(Q)$$
$$T(\lambda P) = \lambda T(P) \quad \text{for } \lambda \in \mathbb{R}.$$

Note:

$$T(x, y) = (x+2y, 3x+4y)$$

can be expressed as matrix multiplication

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+2y \\ 3x+4y \end{pmatrix}$$

we say that

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

represents T.

Theorem Let

$S: \mathbb{R}^2 \to \mathbb{R}^2$ and $T: \mathbb{R}^2 \to \mathbb{R}^2$

be linear transformations

represented by matrices A, B.

Then the linear transformation

$S \circ T : \mathbb{R}^2 \to \mathbb{R}^2, \; v \mapsto S(T(v))$

is represented by the

matrix AB.

<u>Theorem</u> Any linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ can be represented by a matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$.

<u>Proof</u>

well $T(1,0) = (a, b)$ say,

and $T(0,1) = (c,d)$ say.

$$T(x,y) = T(x(1,0) + y(0,1))$$

by linearity
$$= T(x(1,0)) + T(y(0,1))$$
$$= xT(1,0) + yT(0,1)$$
$$= x(a,b) + y(c,d)$$
$$= (ax+cy, bx+dy).$$

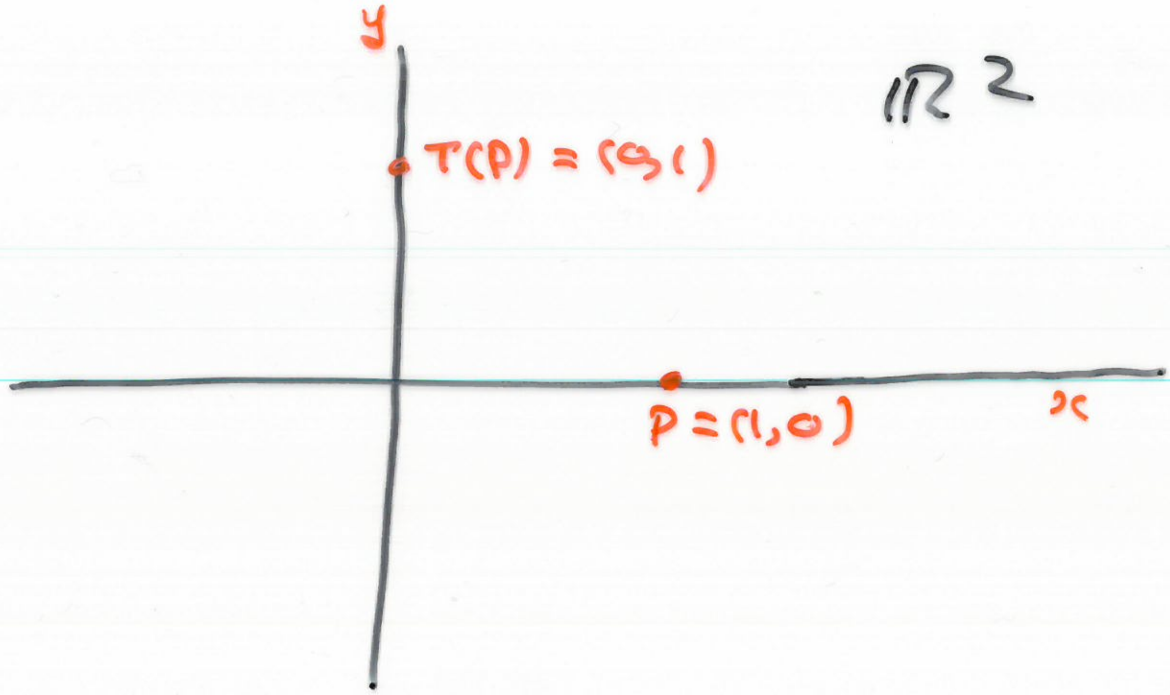
now

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+cy \\ bx+dy \end{pmatrix}$$

FACTS :

- Any reflection in a line through the origin is linear.
- Any rotation about the origin is linear.
- Any composite of linear transformations is linear.

Example Find the matrix representing a reflection in the y-axis, followed by a clockwise rotation of $\frac{5\pi}{2}$ rads about the origin.

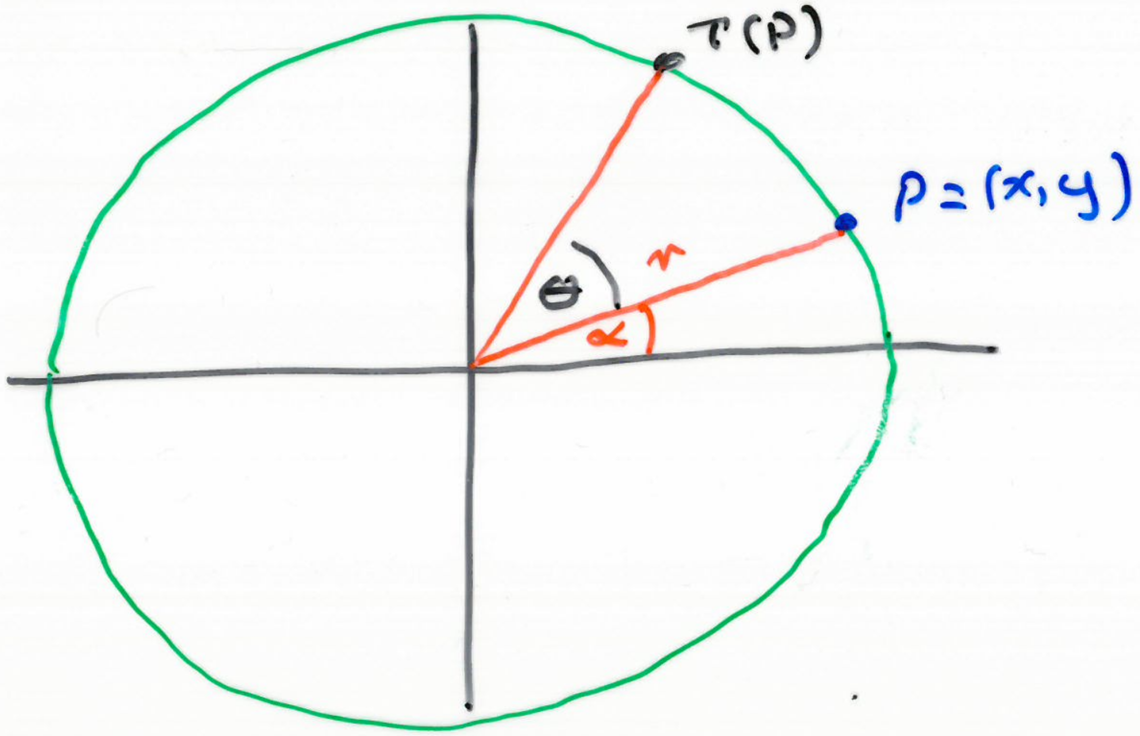


$$\begin{array}{c} a\ \ b \\ \text{So } T(1,0) = (0,1) \\ T(0,1) = (-1,0) \\ c\ \ d \end{array}$$

The required matrix is

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Consider a anticlockwise rotation of the plane through an angle θ about the origin. What matrix represents this transformation.



If $P = (x, y) = (r\cos\alpha, r\sin\alpha)$

then

$T(P) = (r\cos(\theta+\alpha), r\sin(\theta+\alpha))$

$= r(\cos(\alpha+\theta), \sin(\alpha+\theta))$

$$= r(\cos\alpha\cos\theta - \sin\alpha\sin\theta,\ \sin\alpha\cos\theta + \cos\alpha\sin\theta)$$

$$= (x\cos\theta - y\sin\theta,\ y\cos\theta + x\sin\theta)$$

So

$$T(p) = \begin{pmatrix} x\cos\theta - y\sin\theta \\ y\cos\theta + x\sin\theta \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{\text{Matrix of rotation through an angle } \theta \text{ about the origin (anti-clockwise).}} \begin{pmatrix} x \\ y \end{pmatrix}$$