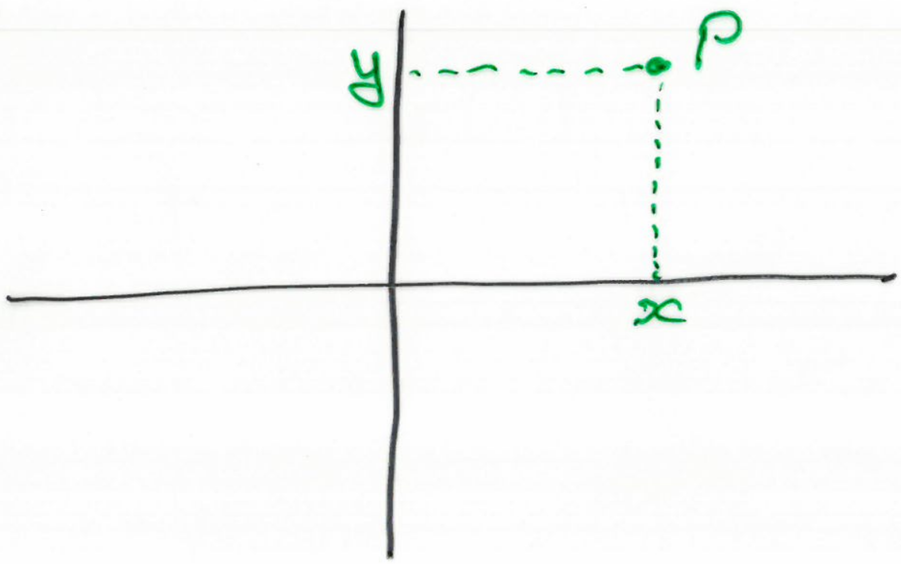
Linear Transformations of the plane

$\mathbb{R}^2$ is the xy-plane



Any point P in the plane can be represented by a pair of real number (x, y).

A transformation of the plane is just a function

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

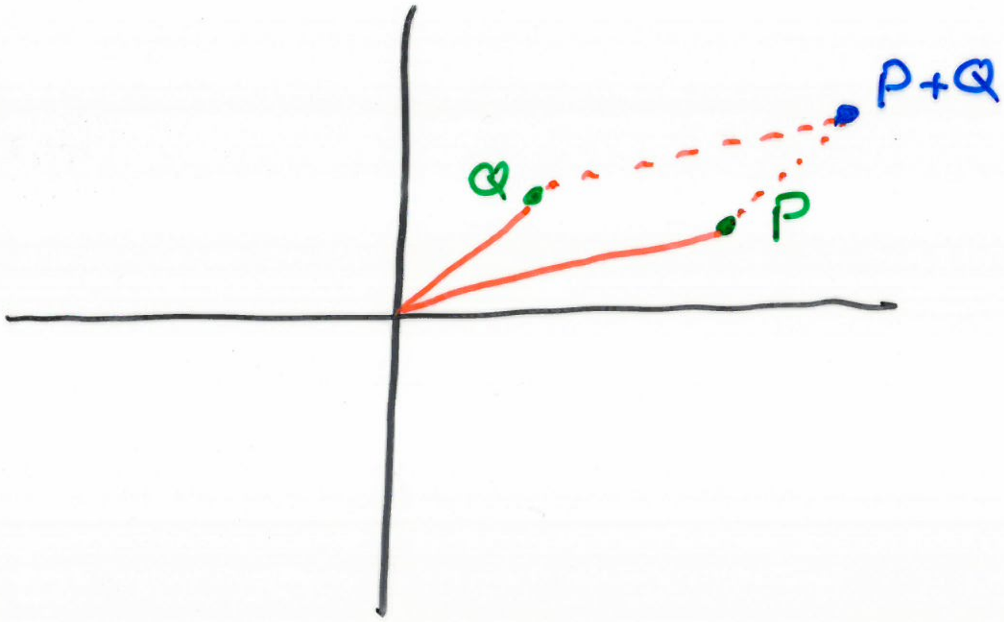
which sends each point $P = (x, y)$ to some point $T(P)$.

We can add two points

$P = (x, y) \qquad Q = (x', y')$

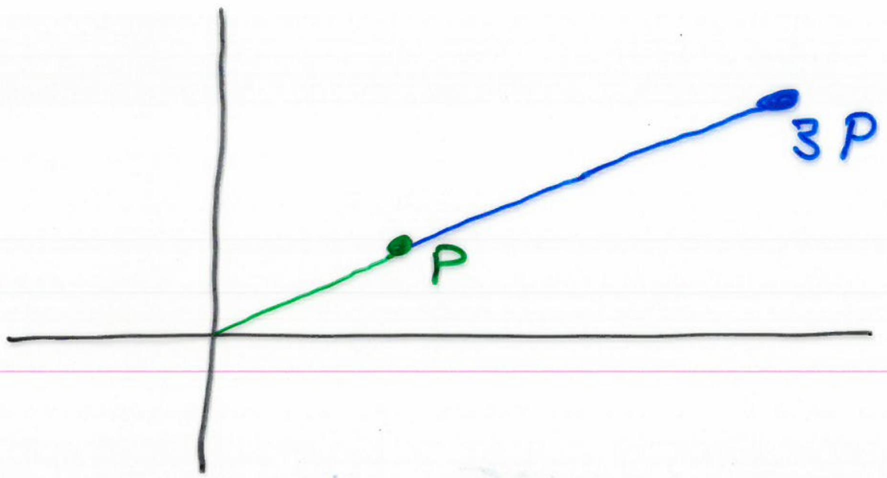
using matrix addition

$$\boxed{P + Q = (x + x', y + y')}.$$



We can multiply a point $P = (x, y)$ by a scalar $\lambda \in \mathbb{R}$ using the formula

$$\boxed{\lambda P = (\lambda x, \lambda y)}$$



Definition A transformation

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

is said to be linear if:

1) $T(P+Q) = T(P) + T(Q)$

2) $T(\lambda P) = \lambda T(P)$

for all $P, Q \in \mathbb{R}^2$, $\lambda \in \mathbb{R}$.

Example Consider the transformation

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$(x,y) \longmapsto (3x+7y,\ 2x+5y)$$

For instance

$$T(1,2) = (17, 12)$$

$$T(-3,1) = (-2, -1)$$

Is T linear?

Consider $P = (x,y)$, $Q = (x', y')$

$$T(P+Q) = T(x+x', y+y')$$

$$= (3(x+x') + 7(y+y'), 2(x+x') + 5(y+y'))$$

$$= (3x+7y+3x'+7y', 2x+5y+2x'+5y')$$

$$= (3x+7y, 2x+5y) + (3x'+7y', 2x'+5y')$$

$$= T(P) + T(Q).$$

Also

$$T(\lambda P) = T(\lambda x, \lambda y)$$

$$= (3\lambda x + 7\lambda y, 2\lambda x + 5\lambda y)$$

$$= \lambda(3x+7y, 2x+5y)$$

$$= \lambda T(P).$$

Thus T is a linear transformation.

<u>Example</u> Consider

$T : \mathbb{R}^2 \to \mathbb{R}^2, \ (x, y) \mapsto (x^2, y^2)$

e.g. $T(1,2) = (1,4)$

Is T linear ?

Consider

$$P = (-1, 1)$$

$$\lambda = 3$$

$$T(\lambda P) = T(-3, 3) = (9,9)$$

$$\lambda T(P) = 3T(-1,1) = 3(1,1) = (3,3)$$

Since $(3,3) \neq (9,9)$

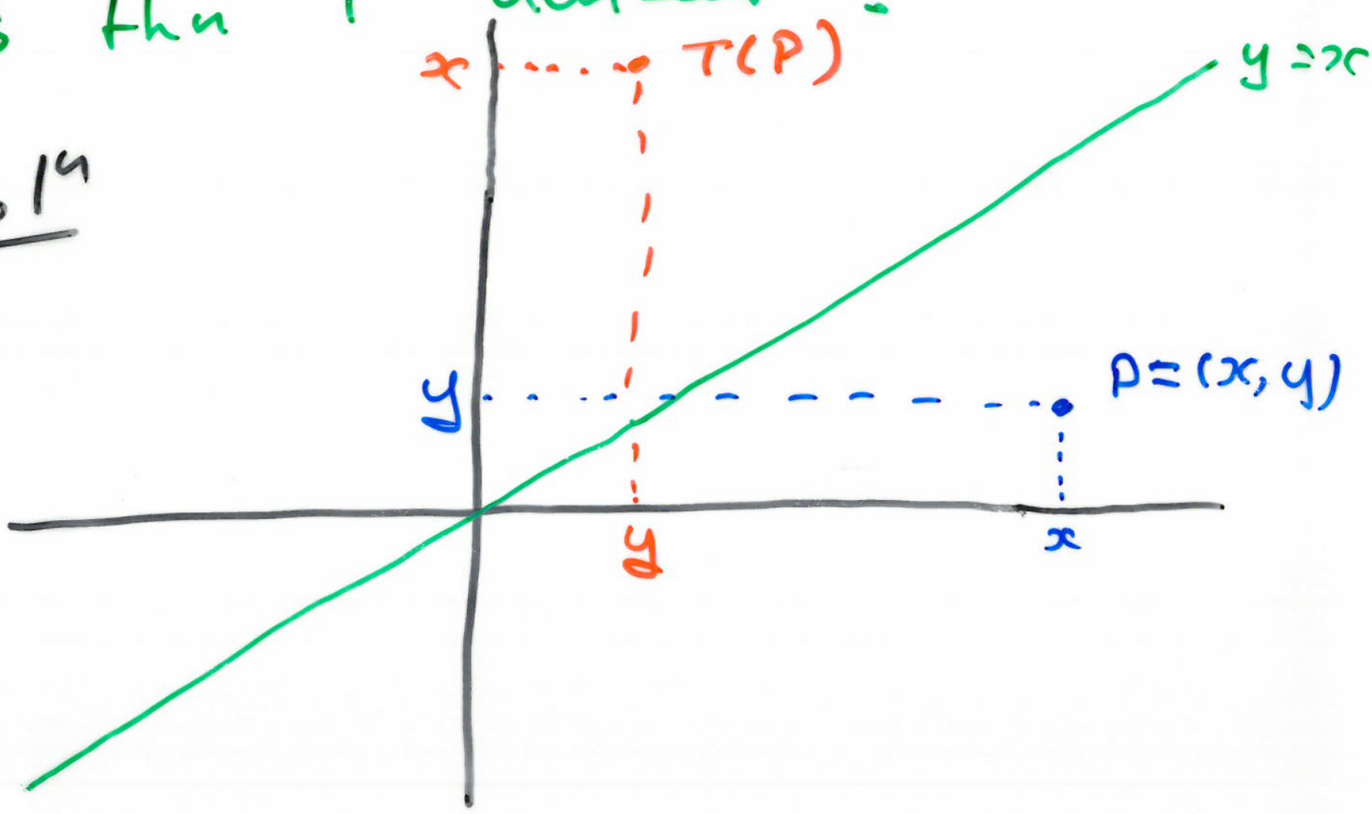
this transformation T is

<u>not</u> linear.

<u>Example</u> Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation obtained by reflecting in the line $y = x$.

Is this T linear?

<u>Sol^n</u>



So $T(x, y) = (y, x)$

Consider $P = (x, y)$, $Q = (x', y')$

$$
\begin{aligned}
T(P+Q) &= T(x+x', y+y') \\
&= (y+y', x+x') \\
&= (y, x) + (y', x') \\
&= T(P) + T(Q)
\end{aligned}
$$

$$
\begin{aligned}
T(\lambda P) &= T(\lambda x, \lambda y) \\
&= (\lambda y, \lambda x) \\
&= \lambda (y, x) \\
&= \lambda T(P).
\end{aligned}
$$

Thus reflection in the line $y = x$ __is__ linear.