Semester I Examinations 2013-14

Exam Code(s)	1BMS1, 1BPT1, 1BS1, 1MR1, 1FM1, 1EH1, 1BME1, 1BA1, 1BA7, 1BDT1, 1BIS1, 1UPA1, 1OA6, 1BCT1
Exam	First Year
Module	MATHEMATICS
Module Code	MA133-1 & MA180-1 & MA185-1 & MA190-1
External Examiner(s)	Dr C. Campbell
Internal Examiner(s)	Dr. G. Pfeiffer Dr. J. Burns Dr. A. McCluskey Dr J. Ward
Instructions	Answer all six questions.
Duration	2 hours
No. of Pages	3 pages (including this cover page)
Discipline	Mathematics

Requirements:

Release to Library:	Yes
Other Materials	Non-programmable calculators

1.

(a) Determine the seventh digit of the ISBN number `0-486-45?62-6`.

(b) Calculate the number $\phi(175)$ of integers from 1 to 175 that are coprime to 175. Use Euler's Theorem to compute

$$3^{1923} \mod 175.$$

(c) Solve the simultaneous congruences:

$$\begin{aligned} x &\equiv 5 \pmod{7}, \\ x &\equiv 3 \pmod{8}, \\ x &\equiv 1 \pmod{9}. \end{aligned}$$

2.

(a) The ciphertext

`EMFIIVZS`

was produced by applying the function

$$f_E\colon \begin{pmatrix} x \\ y \end{pmatrix} \mapsto A \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{with } A = \begin{pmatrix} 6 & 15 \\ 5 & 16 \end{pmatrix}$$

to 2-letter message units over the alphabet $\mathtt{A} = 0$, $\mathtt{B} = 1$, $\ldots$, $\mathtt{Z} = 25$. Calculate $A^{-1} \pmod{26}$ and hence determine the first FOUR letters of plaintext.

(b) Find the inverse of

$$A = \begin{pmatrix} -1 & 0 & 5 \\ -2 & -1 & 8 \\ -1 & 0 & 4 \end{pmatrix}.$$

3.

(a) Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be reflection in the line $y = x$ and let $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ be clockwise rotation through $90°$ around the origin. Find the point $v = (x, y) \in \mathbb{R}^2$ such that $f(g(v)) = (-3, -2)$.

(b) Consider

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \phi_1 = \frac{1+\sqrt{5}}{2}, \quad \phi_2 = \frac{1-\sqrt{5}}{2}.$$

(i) Compute the sum, the difference and the product of ϕ_1 and ϕ_2.

(ii) Verify that

$$v_1 = \begin{pmatrix} -1 \\ \phi_2 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} -1 \\ \phi_1 \end{pmatrix}$$

are eigenvectors of A, and determine the corresponding eigenvalues.

(iii) Find a diagonal matrix D and an invertible matrix T such that $A = TDT^{-1}$.

(iv) Hence solve the recurrence relation

$$f_{n+1} = f_n + f_{n-1}, \quad f_0 = 0, \quad f_1 = 1, \quad n \geqslant 1.$$

4.

(a) For the function $f(x) = \dfrac{x}{x-4}$, determine $\lim_{x\to+\infty} f(x)$, $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to 4^-} f(x)$. Hence write down the horizontal and vertical asymptotes of f.

(b) Calculate the following limits where they exist:

(i) $\displaystyle\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9}$ (ii) $\displaystyle\lim_{x\to -5} \frac{\frac{1}{5}+\frac{1}{x}}{5+x}$ (iii) $\displaystyle\lim_{x\to 1} \frac{|x-1|}{x-1}$.

(c) Explain, with reference to the definition of continuity, why the following function $g : \mathbb{R} \to \mathbb{R}$ is continuous at *each* point of $\mathbb{R}$:

$$g(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

5.

(a) State the Intermediate Value Theorem and use it to prove that the equation

$$\sqrt[3]{x} = 1 - x$$

has exactly one solution in $(0, 1)$.

(b) Consider the function

$$f(x) = \frac{x^2}{\sqrt{x+1}}.$$

(i) Write down the natural domain of f.

(ii) Determine all critical points of f, recalling the domain of f from (i).

(iii) Find the intervals on which f increases/decreases.

(iv) For each critical point, decide whether it is a maximum, a minimum, or neither.

6.

(a) Find antiderivatives of each of the following functions

$$f(t) = e^{5t}, \qquad g(t) = t\sin(t^2), \qquad h(t) = \frac{1}{\sqrt{2t+1}}.$$

(b) It is known that for an ideal pendulum and for small initial displacement angle θ_0 (from the vertical), the displacement angle $\theta(t)$ at time t seconds is described by the differential equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta(t) = 0, \quad \theta(0) = \theta_0, \quad \frac{d\theta}{dt}(0) = 0,$$

where g is acceleration due to gravity and l is the length of the pendulum. By considering the *Cosine* function or otherwise solve the above differential equation. When will the pendulum first return to the vertical?