Clock arithmetic is very much like the school arithmetic. In particular, the following rules hold:

1) $a+b = b+a$ (commutativity)

1') $a \times b = b \times a$

2) $(a+b)+c = a+(b+c)$ (associativity)

2') $(a \times b) \times c = a \times (b \times c)$

3) $a \times (b+c) = a \times b + a \times c$ (distributivity)

We'll now study an example of an arithmetic where 1') does not hold.

Matrix Arithmetic

A matrix is an array of numbers arranged neatly in rows and columns. Each row has the same length. Each column has the same length.

Example

$$\begin{pmatrix} 1 & 2 & 4 \\ -1 & 3 & -2 \end{pmatrix}$$ 2×3 matrix

$$\begin{pmatrix} 1 & 2 \\ 27 & -34 \end{pmatrix}$$ 2×2 matrix

$$\begin{pmatrix} 1 & 2 & 3 & 4 & -5 \end{pmatrix}$$ 1×5 matrix
also a row vector

$$\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$ 3×1 matrix
also a column vector

Matrix Addition

Two $m \times n$ matrices A, B are **added** by adding corresponding entries.

$$\underbrace{\begin{pmatrix} 1 & 2 & -3 \\ 4 & 6 & 0 \end{pmatrix}}_{A} + \underbrace{\begin{pmatrix} -2 & 1 & -3 \\ 1 & 3 & 0 \end{pmatrix}}_{B} = \underbrace{\begin{pmatrix} -1 & 3 & -6 \\ 5 & 9 & 0 \end{pmatrix}}_{A+B}$$

$$\underbrace{\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}}_{A} + \underbrace{\begin{pmatrix} -3 & -2 \\ -1 & 0 \end{pmatrix}}_{B} = \underbrace{\begin{pmatrix} -2 & 0 \\ -2 & 3 \end{pmatrix}}_{A+B}$$

$$\underbrace{\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \end{pmatrix}}_{A} + \underbrace{\begin{pmatrix} 2 & 1 & 0 & -1 & -2 \end{pmatrix}}_{B}$$

$$= \underbrace{\begin{pmatrix} 3 & 3 & 3 & 3 & 3 \end{pmatrix}}_{A+B}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} -1 & -2 & -3 \\ 0 & 1 & 5 \end{pmatrix}$$

Can not be added.

Given a matrix A we write $-A$ to denote the matrix got from A by placing a "$-$" in front of each entry.

$$A = \begin{pmatrix} 2 & 4 \\ 6 & -8 \end{pmatrix}$$

$$-A = \begin{pmatrix} -2 & -4 \\ -6 & 8 \end{pmatrix}$$

Note:

$A + (-A)$ = matrix with entries all zero

$$\begin{pmatrix} 2 & 4 \\ 6 & -8 \end{pmatrix} + \begin{pmatrix} -2 & -4 \\ -6 & 8 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

A matrix whose entries are all zero is called a zero matrix and often denoted by $\mathbb{O}$.

Thus, for any matrix A

$$A + (-A) = \mathbb{O}.$$

Multiplication of a row vector by a column vector

Let

$$R = (a_1, a_2, \dots, a_n)$$

be a row vector of length n.

Let

$$C = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

be a column vector of length n.

We define

$$R.C = (a_1, a_2, \dots, a_n) \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$= a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

<u>example</u>

$$R = (-1, 3, 4) \qquad C = \begin{pmatrix} -2 \\ 0 \\ 5 \end{pmatrix}$$

$$R.C = (-1)(-2) + (3)(0) + (4)(5)$$

$$= 22$$

Matrix Multiplication

A matrix

$$A = \begin{array}{c} \\ R_1 \\ R_2 \end{array} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

can be regarded as a collection of rows R_i

A matrix

$$B = \begin{array}{c} \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} \\ \begin{pmatrix} -1 & 2 & 1 \\ 1 & 3 & 3 \\ 0 & 2 & 0 \end{pmatrix} \end{array}$$

can be regarded as a collection of columns C_j.

Let A be an $m \times n$ matrix.

Let B be a $n \times p$ matrix.

We define the product

$$A \times B$$

as

$$\underbrace{\begin{pmatrix} — & R_1 & — \\ — & R_2 & — \\ & \vdots & \\ — & R_m & — \end{pmatrix}}_{A} \quad \underbrace{\begin{pmatrix} | & | & & | \\ C_1 & C_2 & \cdots & C_p \\ | & | & & | \end{pmatrix}}_{B}$$

$$= \underbrace{\begin{pmatrix} R_1 C_1 & R_1 C_2 & \cdots & R_1 C_p \\ R_2 C_1 & R_2 C_2 & \cdots & R_2 C_p \\ \vdots & & & \vdots \\ R_m C_1 & & & R_m C_p \end{pmatrix}}_{A \times B}$$

Example

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

$$B = \begin{pmatrix} -1 & 2 & 1 \\ 1 & 3 & 3 \\ 0 & 2 & 0 \end{pmatrix}$$

Calculate AB

Sol^n

$$AB = \begin{pmatrix} 1 & 14 & 7 \\ 1 & 35 & 19 \end{pmatrix}$$