Equations

Problem

Solve $3x \equiv 13 \mod 26$

$x \equiv 3^{-1} \cdot 13 \mod 26$

$x \equiv 9 \cdot 13 \mod 26$

$x \equiv (8+1) \cdot 13 \mod 26$

$x \equiv 8 \cdot 13 + 1 \cdot 13 = 13$

Problem

Solve $4x \equiv 12 \mod 26$

~~$x \equiv 4^{-1} \cdot 12 \mod 26$~~

One solution is

$x \equiv 3 \mod 26.$

Chinese Remainder Theorem

Find the smallest ~~positive~~ non negative integer x such that the following equations are simultaneously satisfied:

$$\left.\begin{aligned} x &\equiv 3 \pmod{13} \\ x &\equiv 6 \pmod{14} \\ x &\equiv 9 \pmod{15} \end{aligned}\right\} *$$

Soln

Let $a \equiv 14^{-1} \pmod{13}$

$b \equiv 15^{-1} \pmod{13}$

A first attempt at a solution to $*$ is

$$X = 3 \cdot 14 \cdot a \cdot 15 \cdot b$$

Note:

$$X \equiv 3 \pmod{13}$$

$$X \equiv 0 \pmod{14}$$

$$X \equiv 0 \pmod{15}$$

Let $c = 13^{-1} \pmod{14}$

$d = 15^{-1} \pmod{14}$

Let $y = 6 \cdot 13 \cdot c \cdot 15 \cdot d$

Note:

$$y \equiv 0 \pmod{13}$$

$$y \equiv 6 \pmod{14}$$

$$y \equiv 0 \pmod{15}$$

Now set

$e = 13^{-1} \pmod{15}$

$f = 14^{-1} \pmod{15}$

Let

$$Z = 9.13.e.14.f$$

Note

$$Z \equiv 0 \mod 13$$

$$Z \equiv 0 \mod 14$$

$$Z \equiv 9 \mod 15$$

Now take

$$x = X + Y + Z$$

Note

$$x \equiv X+Y+Z \equiv 3+0+0 \equiv 3 \mod 13$$

$$x \equiv X+Y+Z \equiv 0+6+0 \equiv 6 \mod 14$$

$$x \equiv X+Y+Z \equiv 0+0+9 \equiv 9 \mod 15$$

To finish, we need to calculate X, Y, Z and $X+Y+Z$.

$a \equiv 14^{-1} \mod 13$

$\boxed{a = 1}$

$b \equiv 15^{-1} \mod 13$

$b \equiv 2^{-1} \mod 13$

$\boxed{b = 7}$

$c \equiv 13^{-1} \mod 14$

$c \equiv (-1)^{-1} \mod 14$

$c \equiv -1$

$\boxed{c \equiv 13}$

$d \equiv 15^{-1} \mod 14$

$\boxed{d = 1}$

$e \equiv 13^{-1} \mod 15$

$\boxed{e = 7}$

$f \equiv 14^{-1} \mod 15$

$\boxed{f = 14}$

$X = 3.14.1.15.7$

$Y = 6.13.13.15.14$

$Z = 9.13.7.14.4$

$x = X + Y + Z$

$\equiv 2694 \mod 13.14.15$