MA180 – 2014/15

Q1a)

$37 = 2.14 + 9$

$14 = 9 + 5$

$9 = 5 + 4$

$5 = 4 + 1$ gcd(37, 14)

$$
\begin{aligned}
1 &= 5 - 4 \\
&= 5 - (9 - 5) \\
&= 2.5 - 9 \\
&= 2(14 - 9) - 9 \\
&= 2.14 - 3.9 \\
&= 2.14 - 3(37 - 2.14) \\
&= 8.14 - 3.37 \\
&\equiv 8.14 \pmod{37}
\end{aligned}
$$

So $14^{-1} \equiv 8 \mod 37$

1a ii)

$$f_E : x \rightsquigarrow 14x \rightsquigarrow 14x+20$$

$$f_D : y \rightsquigarrow y-20 \longrightarrow 14^{-1}(y-20)$$

So

$$f_D : \mathbb{Z}_{37} \to \mathbb{Z}_{37}, \quad x \longmapsto 14^{-1}(x-20)$$

$$x \longmapsto 8(x-20)$$

1a iii)

$$H \longleftrightarrow 17$$

$$V \longleftrightarrow 31$$

$$f_D(H) = f_D(17)$$
$$\equiv 8(17-20)$$
$$\equiv 8(-3)$$
$$\equiv -24$$
$$\equiv 13$$
$$= D$$

Deciphered word is

DEED

$$f_V(V) = f_D(31)$$
$$\equiv 8(31-20)$$
$$\equiv 8 \cdot 11$$
$$\equiv 88 \equiv 14 = E$$

2a) $G = 6$

$A = 0$

$$f_E\begin{pmatrix} G \\ A \end{pmatrix} = f_E\begin{pmatrix} 6 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 6 \\ 0 \end{pmatrix} + \begin{pmatrix} 15 \\ 20 \end{pmatrix} \quad \text{mod } 26$$

$$= \begin{pmatrix} 12 \\ 6 \end{pmatrix} + \begin{pmatrix} 15 \\ 20 \end{pmatrix}$$

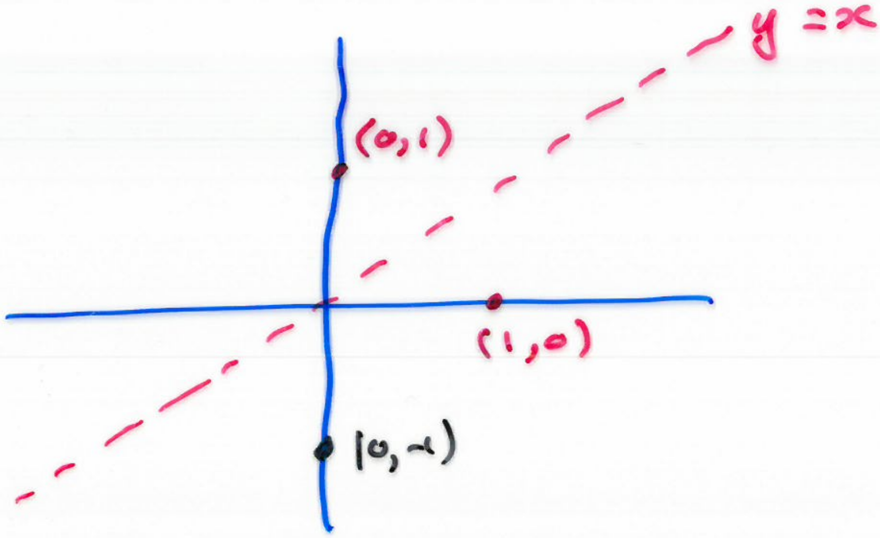
$$\equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} B \\ A \end{pmatrix}$$

Enciphered message :

BABA

3a)



F

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{F}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

G

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{G}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

The matrix of $g \circ f$ is

$$GF = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

3h)

$$x_n = 4x_{n-1} + 5x_{n-2}$$

$$x_{n-1} = x_{n-1}$$

$$\begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix} = \underbrace{\begin{pmatrix} 4 & 5 \\ 1 & 0 \end{pmatrix}}_{A} \begin{pmatrix} x_{n-1} \\ x_{n-2} \end{pmatrix}$$

Let's verify that $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ 1 \end{pmatrix}$ are eigenvalues of A.

$$\begin{pmatrix} 4 & 5 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = -1 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

✓

$\lambda_1 = -1$

$$\begin{pmatrix} 4 & 5 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 25 \\ 5 \end{pmatrix} = 5 \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

✓

$\lambda_2 = 5$

$$E = \begin{pmatrix} 1 & 5 \\ -1 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}$$

6a)

$$\frac{d}{dt}\left(e^{7t}\right) = 7e^{7t}$$

$$\frac{d}{dt}\left(\tfrac{1}{7}e^{7t}\right) = e^{7t}$$

so $\frac{1}{7}e^{7t}$ is an anti-derivative of e^{7t}.

$$\frac{d}{dt}\left(\sin(t^2)\right) = 2t\cos(t^2)$$

so $\frac{1}{2}\sin(t^2)$ is an antider. of $t\cos(t^2)$.

$$\frac{1}{\sqrt{2t+1}} = (2t+1)^{-\frac{1}{2}}$$

$$\frac{d}{dt}\left(2t+1\right)^{\frac{1}{2}} = \tfrac{1}{2}(2t+1)^{-\frac{1}{2}}\,2$$

$= (2t+1)^{-\frac{1}{2}}$ ← anti-derivative of $\frac{1}{\sqrt{2t+1}}$