From last lecture:

Let F_n = number of pairs of rabbits in field after n months.

$F_0 = 1$

$F_1 = 1$

$F_2 = 2$

$F_3 = 3$

$F_4 = 5$

$F_5 = 8$

$F_6 = 13$

21

34

55

89

144

$F_{12} = 233$

$\vdots$

In general

$$F_n = F_{n-1} + F_{n-2}$$

Let's look at

$$F_n / F_{n-1}$$

to see how quickly the rabbit population grows.

$$\frac{1}{1} \quad \frac{2}{1} \quad \frac{3}{2} \quad \frac{5}{3} \quad \cdots\cdots \quad \frac{55}{34}$$

1.618....

$$\frac{89}{55}, \quad \frac{144}{89}, \quad \frac{233}{144}, \quad \cdots\cdots$$

1.618...

Sunflowers!

Maybe the sequence

$$F_n / F_{n-1}$$

Converges as $n \to \infty$?

$$\lim_{n \to \infty} \frac{F_n}{F_{n-1}} = \phi \ ?$$

If the limit ϕ exists then the rabbit population would grow roughly by a factor of ϕ each month.

How do we calculate ϕ ?

$$\boxed{\begin{aligned} F_n &= F_{n-1} + F_{n-2} \\ F_{n-1} &= F_{n-1} \end{aligned}}$$

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{n-1} \\ F_{n-2} \end{pmatrix}$$

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{n-2} \\ F_{n-3} \end{pmatrix}$$

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^3 \begin{pmatrix} F_{n-3} \\ F_{n-4} \end{pmatrix}$$

$$\vdots$$

$$\boxed{\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} F_1 \\ F_0 \end{pmatrix}}$$

If $\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \phi$ then,

for very large n,

$$\frac{F_n}{F_{n-1}} \simeq \phi$$

or

$$\boxed{F_n \simeq \phi F_{n-1}}$$

$$\phi \begin{pmatrix} F_{n-1} \\ F_{n-2} \end{pmatrix} \simeq \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}_{A} \begin{pmatrix} F_{n-1} \\ F_{n-2} \end{pmatrix}$$

So ϕ should be an eigenvalue of A.

To find eigenvalues of A

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 0$$

$$\det\begin{pmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = 0$$

$$(1-\lambda)(-\lambda) - 1 = 0$$

$$\lambda^2 - \lambda - 1 = 0$$

$$\lambda = \frac{+1 \pm \sqrt{1+4}}{2}$$

Eigenvalues of

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

are :

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \approx 1.618\ldots$$

$$\lambda_2 = \frac{1-\sqrt{5}}{2}$$

The number $\boxed{\phi = \frac{1+\sqrt{5}}{2}}$

is called the <u>Golden</u> <u>Ratio</u>