Calculating Eigenvalues & Eigenvectors

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2+2 matrix.

A non-zero vector $v = \begin{pmatrix} x \\ y \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is said to be an eigenvector of A if

$$Av = \lambda v$$

for some number λ. We say that λ is an eigenvalue of A corresponding to v.

To calculate eigenvalues/eigenvectors we need the following:

Proposition Let A be a 2×2 matrix, and let $v = \begin{pmatrix} x \\ y \end{pmatrix}$ be some non-zero vector, and suppose

$$A v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \qquad (*)$$

Then $\det(A) = 0$.

Proof If A^{-1} existed then, from $(*)$, we get

$$A^{-1} A v = A^{-1} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and so

$$v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We conclude that, under the hypothesis of the proposition, A^{-1} does not exist.

Recall the formula

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A).$$

Since A^{-1} does not exist we must have $\det(A) = 0$.

QED.

How can we find the eigenvalues of

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \;?$$

Suppose v is some eigenvector of A. Then

$$Av = \lambda v \qquad (*)$$

for some eigenvalue λ.

$$Av = \lambda I v$$

$$Av - \lambda I v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(A - \lambda I)v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So

$$\det(A - \lambda I) = 0$$

So

$$\det\left(\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) = 0$$

So

$$\det \begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} = 0$$

So

$$(2-\lambda)(2-\lambda) - 1.1 = 0$$

So

$$\lambda^2 - 4\lambda + 3 = 0$$

So

$$(\lambda - 1)(\lambda - 3) = 0$$

Thus $\lambda = 1$ or $\lambda = 3$.
So the eigenvalues of A are $\lambda = 1$ and $\lambda = 3$.

Now let's find ~~the~~ eigenvectors for $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

$\underline{\underline{\lambda = 1}}$ Need $A v = \lambda v$

$$A v = v$$

or need $(A - \lambda I) v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$(A - I) v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

so, for instance,
$$v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
is an eigenvector with eigenvalue $\lambda = 1$.

For $\lambda = 3$

Again, we need

$$(A - \lambda I) v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - 3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So, for instance

$$v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

is an eigenvector for $\lambda = 3$.

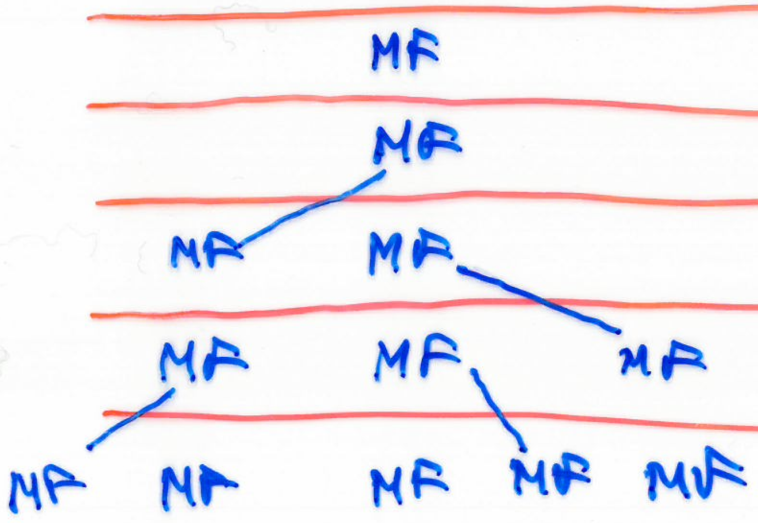
Rabbits

- One newly born male rabbit and one newly born female rabbit are placed in a field.

- Rabbits can mate at the age of 1 month, and one month later the female produces one male/female pair.

- Rabbits never die

- How fast does the rabbit population grow?
- How many rabbits will there be after 100 months?



Months	number of rabbit pairs
0	1
1	1
2	2
3	3
4	5
5	8
6	13
	⋮

At $t=100$ months, how many rabbits are there?