

Why does the Gauss-Jordan method succeed in finding the inverse A^{-1} of a square matrix?

To answer this we need to understand row operations,

Row Operation I

e.g.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 8 & 6 \end{pmatrix} \xrightarrow{R_2 \mapsto R_2 - 2R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 3 & 8 & 6 \end{pmatrix}$$

A B

Also

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 8 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 3 & 8 & 6 \end{pmatrix}$$

E A B

Row operation II

e.g.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 8 & 6 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 8 & 6 \\ 2 & 5 & 5 \end{pmatrix}$$

$\underbrace{\hspace{100px}}_A$ $\underbrace{\hspace{100px}}_B$

Also

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 8 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 8 & 6 \\ 2 & 5 & 5 \end{pmatrix}$$

$\underbrace{\hspace{100px}}_E$ $\underbrace{\hspace{100px}}_A$ $\underbrace{\hspace{100px}}_B$

Row operation III

e.g.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 8 & 6 \end{pmatrix} \xrightarrow{R_2 \mapsto -2R_2} \begin{pmatrix} 1 & 2 & 3 \\ -4 & -10 & -10 \\ 3 & 8 & 6 \end{pmatrix}$$

$\underbrace{\hspace{100px}}_A$ $\underbrace{\hspace{100px}}_B$

Also

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 8 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ -4 & -10 & -10 \\ 3 & 8 & 6 \end{pmatrix}$$

$\underbrace{\hspace{100px}}_E$ $\underbrace{\hspace{100px}}_A$ $\underbrace{\hspace{100px}}_B$

The above establishes:

Result: If A is an $n \times n$ matrix and B is got from A by a row operation

$$A \xrightarrow[\text{operation}]{\text{row}} B$$

then there exists an $n \times n$ matrix E such that

$$B = EA$$

Let's look again at the Gauss-Jordan method for finding A^{-1} .

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$$(A : I) \xrightarrow[\text{operations}]{\text{row}} (I : B)$$

then there are matrices

$E_1, E_2, \dots, E_R$ such that

$$(E_R \dots E_2 E_1) A = I$$

so

$$(E_R \dots E_2 E_1) A A^{-1} = I A^{-1}$$

and

$$(E_R \dots E_2 E_1) I = A^{-1} \quad (*)$$

Note: (*) says $B = A^{-1}$.

This (kind of) proves why the

Gauss-Jordan method works.

Example A factory requires energy, steel and labour to manufacture machines of type A, B and C.

Resource	A	B	C	weekly available
Energy	2 Mwh	3 Mwh	2 Mwh	100 Mwh
steel	1 tonne	1 tonne	4 tonne	70 tonnes
labour	20 hrs	10 hrs	10 hrs	500 hrs

What production figures ensure that all resources are

used?

Soln Let's suppose we manufacture

x units of machine A

y " " B

z " " C

If all resources are used
Then

$$2x + 3y + 2z = 100$$

$$x + y + 4z = 70$$

$$20x + 10y + 10z = 500$$