Yesterday: we checked that the likes of

$T: \mathbb{R}^2 \to \mathbb{R}^2, \ (x,y) \mapsto (x+2y, 3x+4y)$

is linear.

Note that this kind of transformation can be represented as matrix multiplication.

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+2y \\ 3x+4y \end{pmatrix}$$

We say that the matrix

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

represents the transformation T.

<u>Theorem</u> Any linear transformation

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

can be represented by some matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$.

<u>proof</u> suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation.

well

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$$

$$T\begin{pmatrix} x \\ y \end{pmatrix} = T\left(x \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

$$\left.\begin{aligned} &= T\left(x \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) + T\left(y \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \\ &= x\, T\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\, T\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}\right\} \text{uses linearity of } T.$$

$$= x \begin{pmatrix} a \\ b \end{pmatrix} + y \begin{pmatrix} c \\ d \end{pmatrix}$$

$$= \begin{pmatrix} xa \\ xb \end{pmatrix} + \begin{pmatrix} yc \\ yd \end{pmatrix}$$

$$= \begin{pmatrix} xa + yc \\ xb + yd \end{pmatrix} = \begin{pmatrix} ax + cy \\ bx + dy \end{pmatrix}$$

$$= \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Thus T is represented by the matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$. QED

Facts

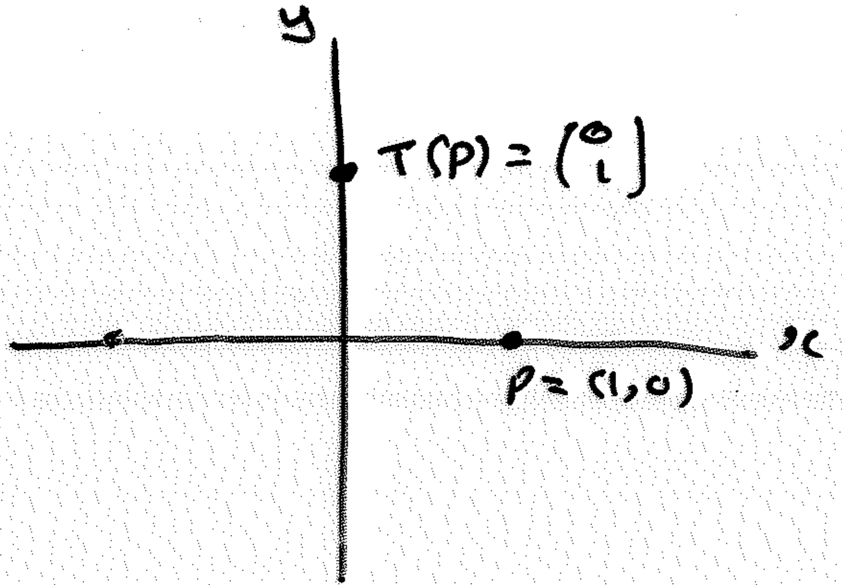
- Any reflection in a line through the origin is linear.
- Any rotation about the origin is linear.

origin $= (0,0)$

- Any composite of linear transformations is a linear transformation.

<u>Example</u> Find the matrix representing a reflection in the y-axis followed by a clockwise rotation of $\frac{5\pi}{2}$ rads about the origin with respect to the standard basis.

<u>Sol</u>



$$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$$

$$T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$$

So T is represented by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$