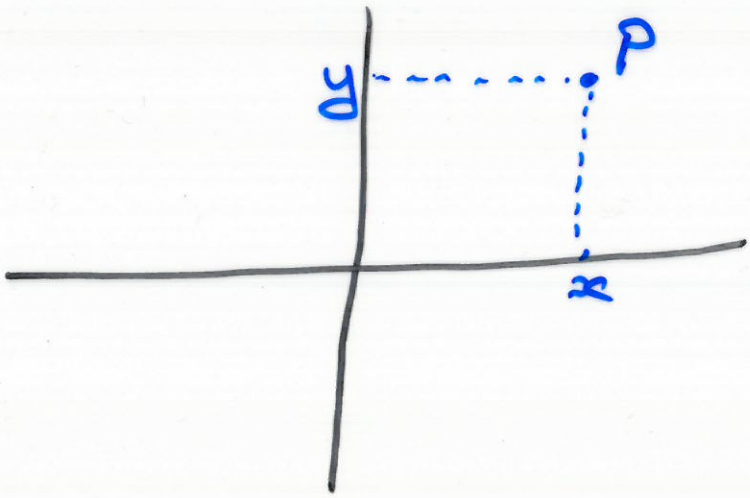
Linear Transformations of the plane

$\mathbb{R}^2$ is the xy-plane



Any point P can be represented by a pair of numbers (x, y).

A transformation of the plane is just a function

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

which sends each point $P = (x, y)$ to some point $T(P)$.

We can add two points

$P = (x, y)$ $\quad Q = (x', y')$

using matrix addition

$$\boxed{P + Q = (x + x', y + y')}.$$

We can multiply a point

$P = (x, y)$ by a scalar

number $\lambda \in \mathbb{R}$ using the

formula

$$\boxed{\lambda P = (\lambda x, \lambda y)}.$$

<u>Definition</u>

Definition A transformation

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

is said to be **linear**

if:

1) $T(P+Q) = T(P) + T(Q)$

2) $T(\lambda P) = \lambda T(P)$

for all $P, Q \in \mathbb{R}^2$, $\lambda \in \mathbb{R}$.

Example Consider the transformation

$$T : \mathbb{R}^2 \to \mathbb{R}^2, \ (x, y) \longmapsto (3x + 7y, \ 2x + 5y).$$

For instance

$$T(1, 2) = (17, 12)$$

$$T(-1, 3) = (18, 13)$$

Is T linear?

Consider $P = (x, y)$, $Q = (x', y')$.

$$T(P+Q) = T(x+x', y+y')$$
$$= \left(3(x+x') + 7(y+y'), 2(x+x') + 5(y+y')\right)$$
$$= (3x + 3x' + 7y + 7y', 2x + 2x' + 5y + 5y')$$
$$= (3x + 7y, 2x + 5y) + (3x' + 7y', 2x' + 5y')$$
$$= T(x, y) + T(x', y')$$
$$= T(P) + T(Q).$$

Also

$$T(\lambda P) = T(\lambda x, \lambda y)$$
$$= (3\lambda x + 7\lambda y, 2\lambda x + 5\lambda y)$$
$$= \lambda(3x + 7y, 2x + 5y)$$
$$= \lambda T(P)$$

Thus T is linear.

Example Consider

$T: \mathbb{R}^2 \to \mathbb{R}^2, \ (x,y) \longmapsto (x^2, y^2)$

e.g. $T(1,2) = (1,4)$

Is T linear ?

Consider $\lambda = 2$

$P = (5,8)$

$$T(\lambda P) = T(10,16) = (100, 256)$$

$$\lambda T(P) = 2T(5,8) = 2(25,64) = (50,128).$$

So, in this instance

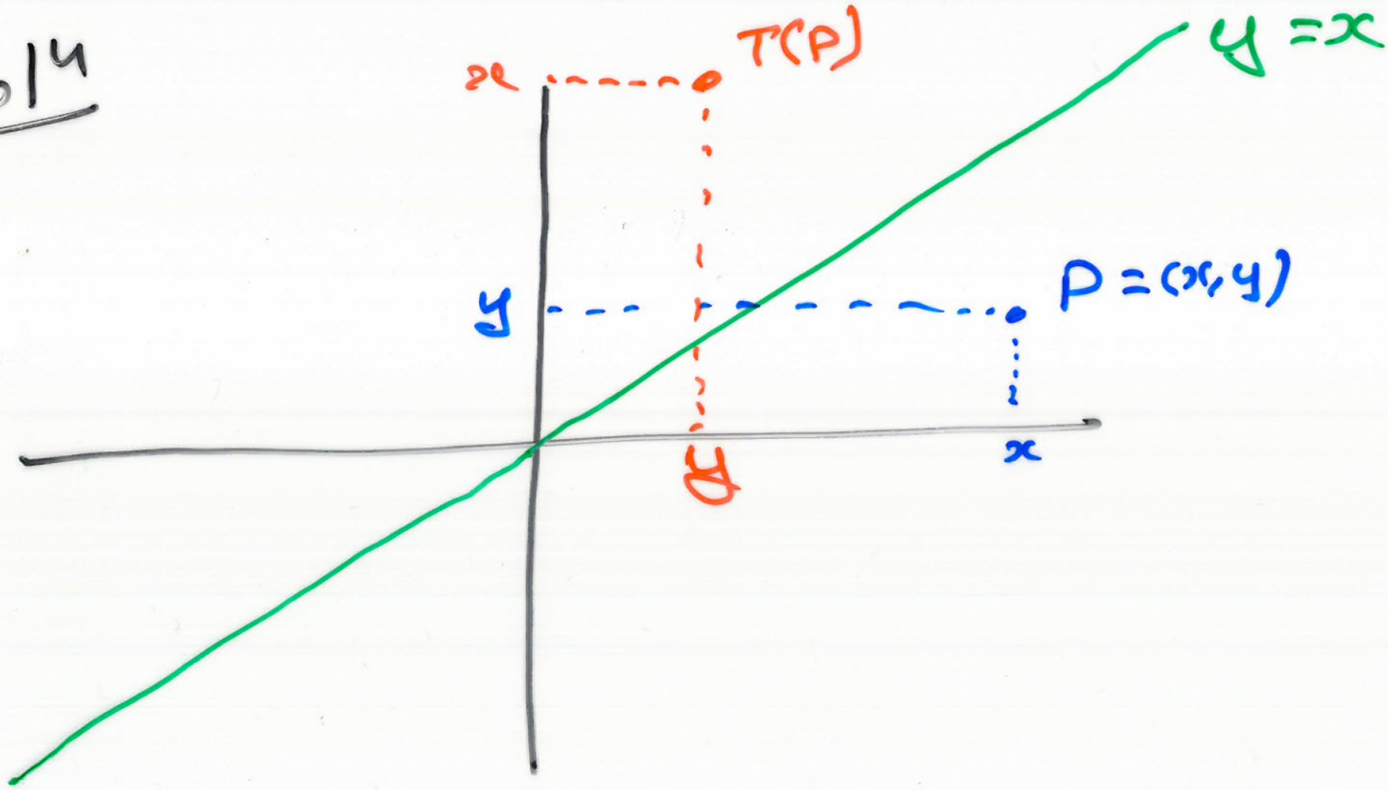
$$T(\lambda P) \neq \lambda T(P)$$

Thus T is <u>not</u> linear.

<u>Example</u> Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation obtained by reflecting in the line $y = x$.

Is this transformation linear?

<u>Soln</u>



So $T(x, y) = (y, x)$.

To see that T is linear consider $P = (x, y)$, $Q = (x', y')$, $\lambda \in \mathbb{R}$.

$$T(P+Q) = T(x+x', y+y')$$
$$= (y+y', x+x')$$
$$= (y, x) + (y', x')$$
$$= T(P) + T(Q).$$

and

$$T(\lambda P) = T(\lambda x, \lambda y)$$
$$= (\lambda y, \lambda x)$$
$$= \lambda (y, x)$$
$$= \lambda T(P).$$

Hence the reflection is linear.