More on Eigenvalues & Eigenvectors

Suppose

$F_0 = 0, \; F_1 = 1, \; F_2 = 1, \; F_3 = 2, \; F_4 = 3, \; F_5 = 5, \; \ldots$

where

$$F_n = F_{n-1} + F_{n-2} .$$

Aim: find an explicit formula for F_n in terms of n but <u>not</u> involving F_{n-1}.

<u>Theorem</u> If a 2×2 matrix A has eigenvalues λ_1, λ_2 with corresponding eigenvectors v_1, v_2 and if the matrix

$$T = \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix}$$

is invertible, then

$$T^{-1} A \, T = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Proof

$$T^{-1} A T = T^{-1} A \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix}$$

$$= T^{-1} \begin{pmatrix} | & | \\ \lambda_1 v_1 , & \lambda_2 v_2 \\ | & | \end{pmatrix}$$

$$= T^{-1} \underbrace{\begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix}}_{T} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$= I \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix} .$$

QED

Example Consider

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

We've seen that A has eigenvalues

$$\phi = \frac{1+\sqrt{5}}{2}$$

$$\bar{\phi} = \frac{1-\sqrt{5}}{2}$$

Let's find corresponding eigenvectors.

Need to solve

$$(A - \lambda I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{for } \lambda = \phi \;\&\; \lambda = \bar{\phi}$$

Note: $\phi\bar{\phi} = -1$

Consider $\lambda = \phi$

$$\begin{pmatrix} 1-\phi & 1 \\ 1 & -\phi \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1-\phi & 1 \\ 1 & -\phi \end{pmatrix} \underbrace{\begin{pmatrix} \phi \\ 1 \end{pmatrix}}_{\substack{\text{eigenvector} \\ \text{for } \lambda = \phi}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Consider $\lambda = \bar{\phi}$

$$\begin{pmatrix} 1-\bar{\phi} & 1 \\ 1 & -\bar{\phi} \end{pmatrix} \underbrace{\begin{pmatrix} 1 \\ -\phi \end{pmatrix}}_{\substack{\text{eigenvector} \\ \text{for } \lambda = \bar{\phi}.}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Set

$$T = \begin{pmatrix} \phi & 1 \\ 1 & -\phi \end{pmatrix}$$

By the above theorem:

$$\boxed{T^{-1} A T = \begin{pmatrix} \phi & 0 \\ 0 & \bar{\phi} \end{pmatrix}}$$

Recall from last lecture

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = A^{n-1} \begin{pmatrix} F_1 \\ F_0 \end{pmatrix}$$

How do we calculate A^{n-1} ?

well

$$\boxed{A = T \underbrace{\begin{pmatrix} \phi & 0 \\ 0 & \bar{\phi} \end{pmatrix}}_{D} T^{-1}}$$

$$\begin{aligned} A^n &= (T D T^{-1})^n \\ &= (T D \cancel{T^{-1}})(\cancel{T} D \cancel{T^{-1}})(\cancel{T} D \cancel{T^{-1}}) \cdots\cdots (\cancel{T} D T^{-1}) \\ &= T D^n T^{-1} \\ &= T \begin{pmatrix} \phi & 0 \\ 0 & \bar{\phi} \end{pmatrix}^n T^{-1} \\ &= T \begin{pmatrix} \phi^n & 0 \\ 0 & \bar{\phi}^n \end{pmatrix} T^{-1} \end{aligned}$$

So

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = A^{n-1} \begin{pmatrix} F_1 \\ F_0 \end{pmatrix}$$

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = T \begin{pmatrix} \phi^{n-1} & 0 \\ 0 & \bar{\phi}^{n-1} \end{pmatrix} T^{-1}$$

$$\begin{pmatrix} A_n \\ F_{n-1} \end{pmatrix} = \frac{1}{-1-\phi^2} \begin{pmatrix} \phi & 1 \\ 1 & -\phi \end{pmatrix} \begin{pmatrix} \phi^{n-1} & 0 \\ 0 & \bar{\phi}^{n-1} \end{pmatrix} \begin{pmatrix} -\phi & -1 \\ -1 & \phi \end{pmatrix}$$

Exercise

$$\boxed{F_n = \frac{1}{\sqrt{5}} \phi^n - \frac{1}{\sqrt{5}} \bar{\phi}^n}$$