Why does the Gauss-Jordan method for finding the inverse $A^{-1}$ of an $n \times n$ matrix $A$ work?

To answer this we need

**Claim** If $A$ is an $n \times n$ matrix and $B$ is got from $A$ by a row operation

$$A \xrightarrow[\text{operation}]{\text{row}} B$$

then there exists an $n \times n$ matrix $E$ such that

$$B = EA.$$

## Row operation I

e.g.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 8 & 6 \end{pmatrix} \xrightarrow{R_2 \longmapsto R_2 - 2R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 3 & 8 & 6 \end{pmatrix}$$

$A$ $\qquad\qquad$ $B$

Also

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 8 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 3 & 8 & 6 \end{pmatrix}$$

$E$ $\qquad$ $A$ $\qquad$ $B$

## Row operation II

e.g.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 8 & 6 \end{pmatrix} \xrightarrow{R_2 \longleftrightarrow R_3} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 8 & 6 \\ 2 & 5 & 5 \end{pmatrix}$$

$A$ $\qquad\qquad$ $B$

Also

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 8 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 8 & 6 \\ 2 & 5 & 5 \end{pmatrix}$$

$$E \qquad\qquad A \qquad\qquad B$$

## Row operation III

e.g.

$$\underset{A}{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 8 & 6 \end{pmatrix}} \xrightarrow{R_2 \mapsto -2R_2} \underset{B}{\begin{pmatrix} 1 & 2 & 3 \\ -4 & -10 & -10 \\ 3 & 8 & 6 \end{pmatrix}}$$

Also

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 8 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ -4 & -10 & -10 \\ 3 & 8 & 6 \end{pmatrix}$$

These examples justify the above claim.

Let's look at the Gauss-Jordan method for finding $A^{-1}$.

If

$$(A : I) \xrightarrow[\text{operations}]{\text{row}} (I : B)$$

then there are matrices

$E_1, E_2, \ldots, E_k$ such that

$$(E_k \cdots E_2 E_1) A = I$$

so

$$(E_k \cdots E_2 E_1) A A^{-1} = I A^{-1}$$

and

$$(E_k \cdots E_2 E_1) I = A^{-1} \quad (*)$$

Note: $(*)$ says $B = A^{-1}$.

This (kind of) proves why the Gauss-Jordan method works.

Example A factory requires energy, steel and labour to manufacture three machines A, B, C.

| Resource | A | B | C | weekly available |
|---|---|---|---|---|
| Energy | 2 Mwh | 3 Mwh | 2 Mwh | 100 Mwh |
| steel | 1 tonne | 1 tonne | 4 tonne | 70 tonne |
| labour | 20 hrs | 10 hrs | 10 hrs | 500 hrs |

What production figures ensure that all resources are used?

Sol$^n$ Let's suppose we manufacture

$x$ units of machine A

$y$ " B

$z$ " C

If all resources are used then

$$\left.\begin{aligned} \boxed{2}x + 3y + 2z &= 100 \\ x + y + 4z &= 70 \\ 20x + 10y + 10z &= 500 \end{aligned}\right\} \text{system of linear equats}$$

The system is equivalent to the following system:

$$\begin{bmatrix} R_2 \longmapsto R_2 - \frac{1}{2} R_1 \\ R_3 \longmapsto R_3 - 10 R_1 \end{bmatrix}$$

$$\begin{aligned} 2x + 3y + 2z &= 100 \\ \boxed{-\tfrac{1}{2}}y + 3z &= 20 \\ -20y - 10z &= -500 \end{aligned}$$

This system is equivalent to

$$[\, R_3 \longrightarrow R_3 - 40 R_2 \,]$$

$$
\begin{aligned}
2x + 3y + 2z &= 100 \\
-\tfrac{1}{2}y + 3z &= 20 \\
-130z &= -1300
\end{aligned}
$$

Back ~~s~~ substitution:

$z = 10$

$y = 20$

$x = 10$

<u>Notation:</u>

2 is the <u>pivot</u> in the first stage

$-\frac{1}{2}$ " " " second stage

The above procedure for solving a system of linear equations is called <u>Gaussian elimination</u>.

When could this procedure fail?

Answer: If one of the pivots is zero.