Frogs

Problem The population of frogs on an island is infected with a disease. Each day 20% of healthy frogs become ill, and 30% of ill frogs become healthy.

There are 500 frogs on the island, of which 100 are initially infected.

Determine the number of infected frogs after 1, 2, ... days, and investigate what happens long term.

Let

x_n = number of healthy frogs on day n

y_n = " " ill " " "

$x_0 = 400$

$y_0 = 100$

$$x_n = 0.8\, x_{n-1} + 0.3\, y_{n-1}$$

$$y_n = 0.2\, x_{n-1} + 0.7\, y_{n-1}$$

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \underbrace{\begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}}_{A} \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix} \begin{pmatrix} 400 \\ 100 \end{pmatrix} = \begin{pmatrix} 350 \\ 150 \end{pmatrix}$$

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix} \begin{pmatrix} 350 \\ 150 \end{pmatrix} = \begin{pmatrix} 325 \\ 175 \end{pmatrix}$$

$$\vdots$$

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = A^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

To investigate $n = 2, 3, 4, \ldots$ let's find eigenvalues of A.

$$A = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}$$

$$\det(A - \lambda I) = 0$$

$$\det \begin{pmatrix} 0.8 - \lambda & 0.3 \\ 0.2 & 0.7 - \lambda \end{pmatrix} = 0$$

$$(0.8 - \lambda)(0.7 - \lambda) - (0.2)(0.3) = 0$$

$$\lambda^2 - 1.5\lambda + 0.5 = 0$$

$$(\lambda - 1)(\lambda - \tfrac{1}{2}) = 0$$

Eigenvalues for A are

$\lambda_1 = 1$ and $\lambda_2 = \frac{1}{2}$.

From the theorem of last lecture

$$T^{-1} A T = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

where T is the 2×2 matrix whose columns are eigenvectors for λ_1 and λ_2.

So

$$A = T \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} T^{-1}$$

$$A^n = T \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}^n T^{-1}$$

$$A^n = T \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2^n} \end{pmatrix} T^{-1}$$

for large n we roughly have

$$A^n = T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} T^{-1}$$

For large n we roughly have

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = A^n \begin{pmatrix} 400 \\ 100 \end{pmatrix} = T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} T^{-1} \begin{pmatrix} 400 \\ 100 \end{pmatrix}$$

and

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = A \underbrace{A^n \begin{pmatrix} 400 \\ 100 \end{pmatrix}} = \underbrace{A^n \begin{pmatrix} 400 \\ 100 \end{pmatrix}}$$

eigenvector of A corresponding to $\lambda_1 = 1$.

Let's find this eigenvector:

$$(A - \lambda_1 I) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -0.2 & 0.3 \\ 0.2 & -0.3 \end{pmatrix} \begin{pmatrix} 300 \\ 200 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Conclusion:

In the long term about 300 frogs will be healthy on any given day.