

More on Eigenvalue & vectors

Suppose

$$F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2, \dots$$

where

$$F_n = F_{n-1} + F_{n-2}$$

Ans: Find an explicit formula for F_n in terms of n but not involving F_{n-1} .

Theorem If a 2×2 matrix A has eigenvalues λ_1, λ_2 with corresponding eigenvectors v_1, v_2 and if the matrix

$$T = \begin{pmatrix} v_1 & v_2 \\ 1 & 1 \end{pmatrix}$$

is invertible, then

$$T^{-1} A T = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

Proof

$$T^{-1}AT = T^{-1}A \begin{pmatrix} 1 & 1 \\ v_1 & v_2 \\ 1 & 1 \end{pmatrix}$$

$$= T^{-1} \begin{pmatrix} 1 & 1 \\ \lambda_1 v_1 & \lambda_2 v_2 \\ 1 & 1 \end{pmatrix}$$

$$= T^{-1} \underbrace{\begin{pmatrix} 1 & 1 \\ v_1 & v_2 \\ 1 & 1 \end{pmatrix}}_T \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$= I \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

QED

Example Consider

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

We've seen that A have eigenvalues

$$\phi = \frac{1 + \sqrt{5}}{2}$$

$$\bar{\phi} = \frac{1 - \sqrt{5}}{2}.$$

Let's find corresponding eigenvectors.
Need to solve

$$(A - \lambda I) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{for } \lambda = \phi \text{ and } \lambda = \bar{\phi}$$

Note: $\phi \bar{\phi} = -1$ ✓

Consider $\lambda = \phi$

$$\begin{pmatrix} 1 - \phi & 1 \\ 1 & -\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1-\sqrt{3}}{2} & 1 \\ 1 & -\phi \end{pmatrix} \begin{pmatrix} \phi \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Eigenvector
for $\lambda = \phi$

Consider $\lambda = \bar{\phi}$.

$$\begin{pmatrix} 1-\bar{\phi} & 1 \\ 1 & -\bar{\phi} \end{pmatrix} \begin{pmatrix} 1 \\ -\phi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Eigenvector
for $\lambda = \bar{\phi}$

Set

$$T = \begin{pmatrix} \phi & 1 \\ 1 & -\phi \end{pmatrix}.$$

By the theorem above

$$T^{-1} A T = \begin{pmatrix} \phi & 0 \\ 0 & \bar{\phi} \end{pmatrix}$$

Alternatively

$$A = T \underbrace{\begin{pmatrix} \phi & 0 \\ 0 & \bar{\phi} \end{pmatrix}}_D T^{-1}.$$

$$A^n = (T D T^{-1})^n$$

$$= (\cancel{T} D \cancel{T^{-1}}) (\cancel{T} D \cancel{T^{-1}}) (\cancel{T} D \cancel{T^{-1}}) \dots (\cancel{T} D \cancel{T^{-1}})$$

$$= T D D D \dots D T^{-1}$$

$$= T D^n T^{-1}.$$

$$= T \begin{pmatrix} \phi & 0 \\ 0 & \bar{\phi} \end{pmatrix}^n T^{-1}$$

$$= T \begin{pmatrix} \phi^n & 0 \\ 0 & \bar{\phi}^n \end{pmatrix} T^{-1}$$

from last lecture

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = A^{n-1} \begin{pmatrix} F_1 \\ F_0 \end{pmatrix}$$

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = T \begin{pmatrix} \phi^{n-1} & 0 \\ 0 & \bar{\phi}^{n-1} \end{pmatrix} T^{-1}$$

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \frac{1}{-1-\phi^2} \begin{pmatrix} \phi & 1 \\ 1 & -\phi \end{pmatrix} \begin{pmatrix} \phi^{n-1} & 0 \\ 0 & \bar{\phi}^{n-1} \end{pmatrix} \begin{pmatrix} -\phi & -1 \\ -1 & \phi \end{pmatrix}$$

Exercise :

$$F_n = \frac{1}{\sqrt{5}} \phi^n - \frac{1}{\sqrt{5}} \bar{\phi}^n$$