Matrix Multiplication

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 8 & 6 \end{pmatrix} \qquad B = \begin{pmatrix} 10 & -12 & 5 \\ -3 & 3 & -1 \\ -1 & 2 & -1 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

So $AB = I$, the identity.

So

$$A^{-1} = B = \begin{pmatrix} 10 & -12 & 5 \\ -3 & 3 & -1 \\ -1 & 2 & -1 \end{pmatrix}$$

Inverting a matrix

How can we find the inverse of a matrix such as

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 8 & 6 \end{pmatrix} \; ?$$

Gauss-Jordan method for finding the inverse of A

$$(A \,\vdots\, I) \xrightarrow[\text{operations}]{\text{row}} (I \,\vdots\, B)$$

Then $B = A^{-1}$.

There are three kinds of row operations:

(I) $R_i \longmapsto R_i + \lambda R_j \quad (j \neq i)$

(II) $R_i \longleftrightarrow R_j$

(III) $R_i \longrightarrow \lambda R_i \quad (\lambda \neq 0)$

Let's illustrate on above matrix A

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 5 & 0 & 1 & 0 \\ 3 & 8 & 6 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 2 & -3 & -3 & 0 & 1 \end{array}\right) \xrightarrow{R_3 \to R_3 - 2R_2}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -1 & 1 & -2 & 1 \end{array}\right) \xrightarrow{R_3 \to -R_3}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right) \xrightarrow[R_2 \to R_2 + R_3]{R_1 \to R_1 - 3R_3}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 4 & -6 & 3 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right) \xrightarrow{R_1 \to R_1 - 2R_2}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 10 & -12 & 5 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right)$$

Hence

$$A^{-1} = \begin{pmatrix} 10 & -12 & 5 \\ -3 & 3 & -1 \\ -1 & 2 & -1 \end{pmatrix}.$$

Towards: why does this method work.

Operation I

e.g.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 8 & 6 \end{pmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 3 & 8 & 6 \end{pmatrix}$$

Also

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 8 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 3 & 8 & 6 \end{pmatrix}$$

In general

$$A \xrightarrow{R_i \to R_i + \lambda R_j} B$$

then

$$E_{ij}^{\lambda} A = B$$

where E_{ij}^{λ} has: 1 on the diagonals

: λ in ith row & jth column

: 0 elsewhere.