Last week: we checked that

$T: \mathbb{R}^2 \to \mathbb{R}^2, (x,y) \mapsto (x+2y, 3x+4y)$

is linear.

Note that

$T(x,y) = (x+2y, 3x+4y)$

can be represented as matrix multiplication

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+2y \\ 3x+4y \end{pmatrix}$$

We say that

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

represents the transformation T.

<u>Theorem</u> Any linear transformation T can be represented by a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Proof

Suppose $T: \mathbb{R}^2 \to \mathbb{R}^2$ is some linear transformation.

well

$T(1,0) = (a,b)$ say

$T(0,1) = (c,d)$ say

Then

$T(x,y) =$

$= T(x(1,0) + y(0,1))$

$$\left.\begin{aligned} &= T(x(1,0)) + T(y(0,1)) \\ &= xT(1,0) + yT(0,1) \end{aligned}\right\} \text{by linearity of } T$$

$= x(a,b) + y(c,d)$

$= (xa, xb) + (yc, yd)$

$= (ax + cy, bx + dy)$.

now

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+cy \\ bx+dy \end{pmatrix}$$

FACTS

- Any reflection in a line through the origin $(0,0)$ is linear.
- Any rotation about the origin is linear.
- Any composition of two linear transformations is linear.

Aside

Given a 2×2 matrix

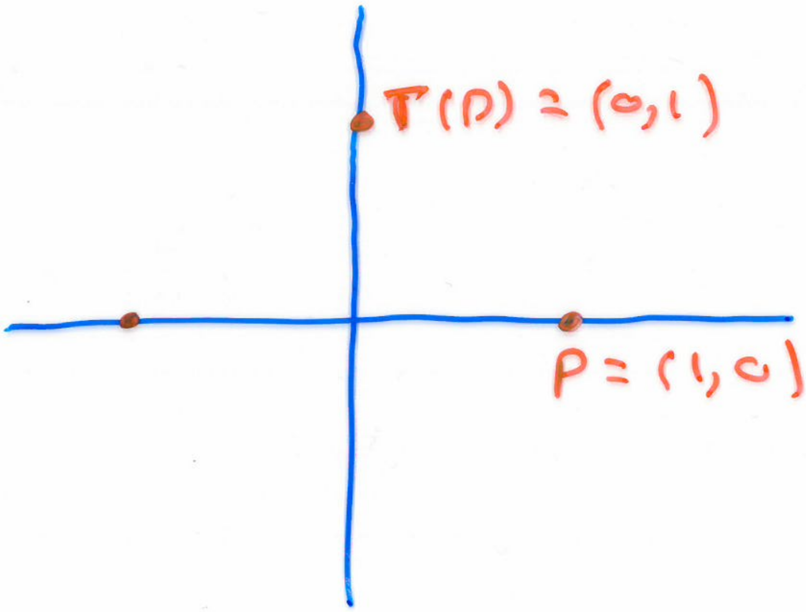
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we define it trace to be the number

$$\mathrm{Tr}(A) = a + d$$

Example Find the matrix representing a reflection in the y-axis, followed by a clockwise rotation of $\frac{5\pi}{2}$ rads about the origin.

Sol^n



$$T(1,0) = (0,1)$$

$$T(0,1) = (1,0)$$

So the required matrix is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Theorem

Let $S : \mathbb{R}^2 \to \mathbb{R}^2$, $T : \mathbb{R}^2 \to \mathbb{R}^2$ be linear transformations represented by matrices A, B. The the linear transformation

$$T \circ S : \mathbb{R}^2 \to \mathbb{R}^2, \quad v \mapsto T(S(v))$$

$$(x,y) \mapsto T(S(x,y))$$

is represented by the matrix

$$BA,$$