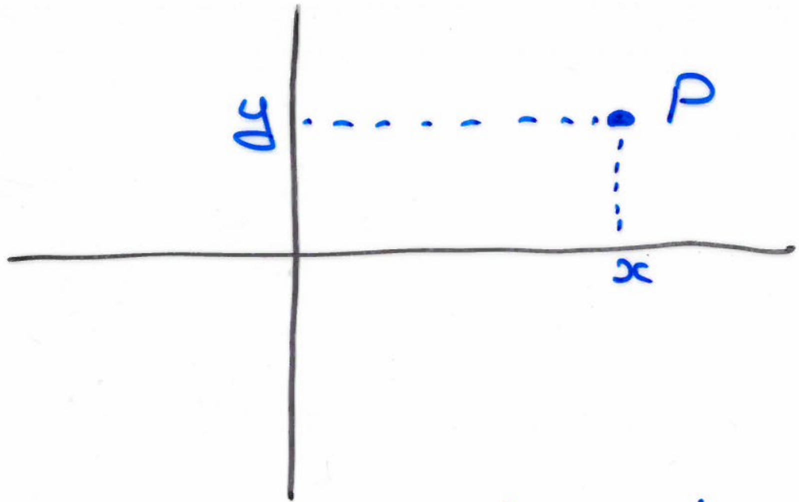
Linear transformations of the plane

$\mathbb{R}^2$ is the xy-plane



Any point P in the plane can be represented by a pair of numbers

$$P = (x, y) .$$

A transformation of the plane is just a function

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad P \longmapsto T(P)$$

which sends each point P to some point $T(P)$.

we can add two points

$$P = (x, y) \qquad Q = (x', y')$$

using matrix addition

$$P + Q = (x + x', y + y').$$

we can multiply a point $P = (x, y)$ by a scalar number $\lambda \in \mathbb{R}$ using

$$\lambda P = (\lambda x, \lambda y).$$

<u>Definition</u> A transformation

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

is said to be <u>linear</u> if

i) $T(P + Q) = T(P) + T(Q)$

ii) $T(\lambda P) = \lambda T(P)$

for all $P, Q \in \mathbb{R}^2$, $\lambda \in \mathbb{R}$.

Example Consider the transformation

$$T : \mathbb{R}^2 \to \mathbb{R}^2, \ (x, y) \longmapsto (x+2y, \ 3x+4y).$$

so

$$T(5, 6) = (17, 39)$$

Is T linear ?

Consider $P = (x, y)$, $Q = (x', y')$

$$T(P+Q) = T(x+x', \ y+y')$$

$$= \Big(x+x'+2(y+y') \ , \ 3(x+x') + 4(y+y') \Big)$$

$$= (x+2y + x'+2y', \ 3x+4y + 3x'+4y')$$

$$= (x+2y, \ 3x+4y) + (x'+2y', \ 3x'+4y')$$

$$= T(P) + T(Q)$$

Also:

$$T(\lambda P) = T(\lambda(x,y))$$

$$= T(\lambda x, \lambda y)$$

$$= (\lambda x + 2\lambda y, 3\lambda x + 4\lambda y)$$

$$= \lambda(x+2y, 3x+4y)$$

$$= \lambda T(P).$$

Since this holds for all λ and all points $P=(x,y)$, $Q=(x',y')$ we see that T is linear.

<u>Example</u> Consider

$T: \mathbb{R}^2 \to \mathbb{R}^2, (x,y) \longmapsto (x^2, y^2)$

Lets consider $\lambda = 4$, $P = (7, 8)$.

$T(\lambda P)$

$= T(4 \cdot (7, 8))$

$= T(28, 32)$

$= (784, 1024)$.

$\lambda T(P) = 4\, T(7, 8)$

$= 4\,(49, 64)$

$= (196, 256)$.

So $T(\lambda P) \neq \lambda T(P)$ for some λ and P.

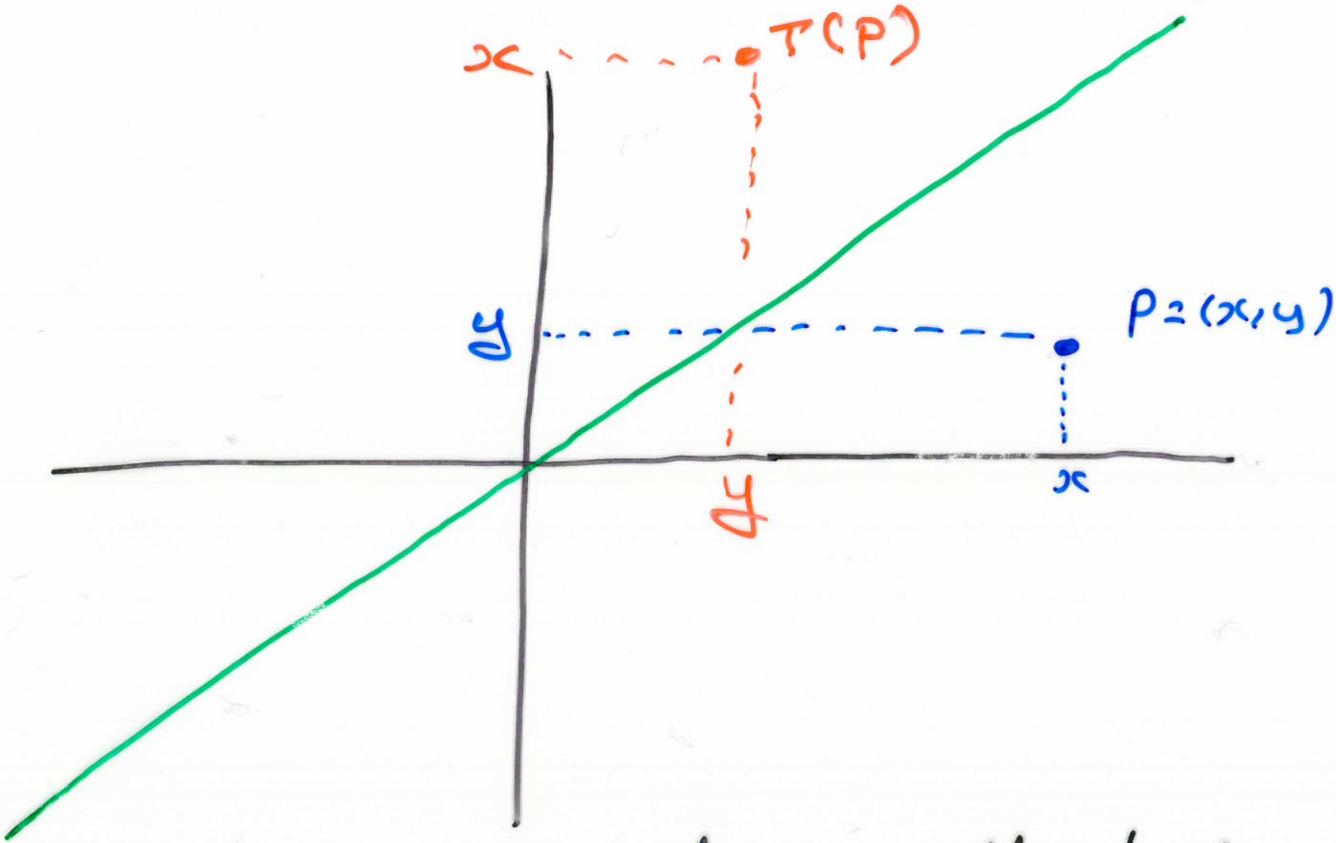
Hence T is not linear.

Example Let

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$

be the transformation obtained by reflecting in the line $y = x$.

Is this transformation linear?



So a formula for reflection is

$$T : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto (y, x)$$

Is reflection linear ?

For $P = (x, y)$, $Q = (x', y')$

we have

$$T(P+Q) = T(x+x', y+y')$$

$$= (y+y', x+x')$$

$$= (y, x) + (y', x')$$

$$= T(x,y) + T(x', y')$$

$$= T(P) + T(Q).$$

For $\lambda \in \mathbb{R}$

$$T(\lambda P) = T(\lambda x, \lambda y)$$

$$= (\lambda y, \lambda x)$$

$$= \lambda (y, x)$$

$$= \lambda T(x, y)$$

$$= \lambda T(P).$$

Hence reflection is linear.