## SQUEEZE THEOREM ( a.k.a. Sandwich Lemma )

Suppose $f(x) \leq g(x) \leq h(x)$ near $a$ ( except possibly at $x=a$ )

and
$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L.$$

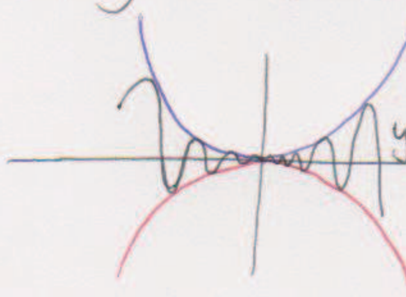
Then $\lim_{x\to a} g(x) = L$ as well.

<u>Example</u>. Calculate $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$, if it exists.

Since ~~sin~~ $-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$, we have $\overset{f(x)}{-x^2} \leq x^2 \sin\left(\frac{1}{x}\right) \leq \overset{h(x)}{x^2}$

Since $\lim_{x\to 0}(-x^2) = \lim_{x\to 0}(x^2) = 0$, by the Squeeze Theorem,

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0.$$




## INFINITE LIMITS.

Suppose $f$ is defined ~~at~~ near $a$, except possibly at $x = a$.

Then $\lim_{x\to a} f(x) = +\infty$ <u>means</u>

that the values of $f(x)$ can be made as large and positive as we want by taking $x$ sufficiently close to $a$, but not equal to $a$.

Define $\lim_{x\to a} f(x) = -\infty$ by writing negative instead of positive above.

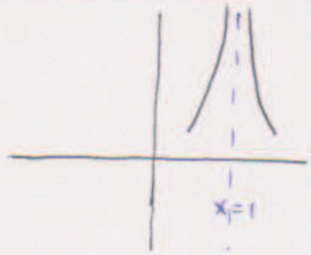
**Example.** $\lim_{x\to 0} \frac{1}{x^2} = +\infty$. and $\lim_{x\to 3} \frac{-2}{(x-3)^2} = -\infty$.

**Example.** $f(x) = \frac{1}{x}$ satisfies that $\lim_{x\to 0^+} \frac{1}{x} = +\infty$ but $\lim_{x\to 0^-} \frac{1}{x} = -\infty$.

Whenever $\lim_{x\to a^\pm} f(x) = \pm\infty$ we say that $f$ has a vertical asymptote at $x = a$.

**Example.** $f(x) = \frac{1}{(x-1)^2}$

$\lim_{x\to 1} f(x) = +\infty$.




$x = 1$ is a vertical asymptote for $f(x) = \frac{1}{x^2}$.

**LIMITS AT INFINITY.** (section 4.4 of Stewart's book)

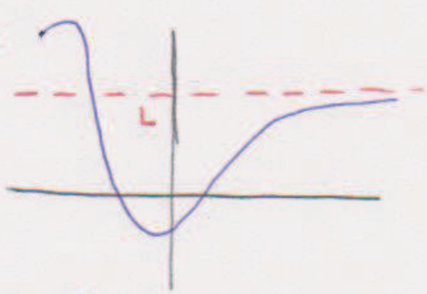
Suppose $f$ is a function defined on some interval $(c, +\infty)$. Then

$$\lim_{x\to +\infty} f(x) = L$$

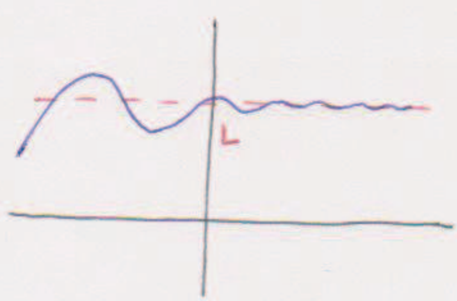
means that the values of $f(x)$ can be made arbitrarily close to $L$ by taking $x$ sufficiently large and positive

( Define $\lim_{x\to -\infty} f(x) = L$ by writing negative instead of positive above. )

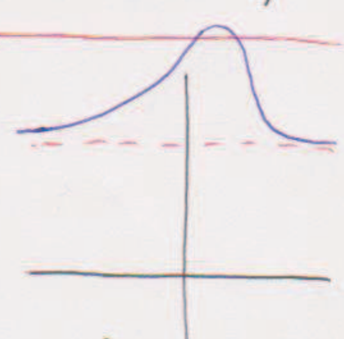
**Ex.**




$\lim_{x\to +\infty} f(x) = L$




$\lim_{x\to +\infty} f(x) = L$



$\lim_{x\to +\infty} f(x) = L$

$\lim f(x) = L$.

**Example.** $\lim_{x\to+\infty} \frac{1}{x^2} = \lim_{x\to-\infty} \frac{1}{x^2} = +\infty.$

$\lim_{x\to+\infty} \frac{1}{2x^3} = 0.$ ; $\lim_{x\to-\infty} \frac{1}{4x^5} = 0$

(In general, $\lim_{x\to\pm\infty} \frac{1}{x^r} = 0$ (for $r$ rational)) for which $x^r$ is defined.

Whenever $\lim_{x\to\pm\infty} f(x) = L$ we say that $y = L$ is a horizontal asymptote for $f$.

SOME INDETERMINACIES

A. Case $\frac{k}{0}$, $k \neq 0$.: Lateral limits at the point will be $+\infty$ or $-\infty$.

Ex. $\lim_{x\to1^+} \frac{1}{x-1} = +\infty$ ; $\lim_{x\to2^-} \frac{1}{x-2} = -\infty.$

B. Case $\frac{0}{0}$. If a rational function (quotient), ~~[illegible]~~ simplify numerator and denominator

Ex. $\lim_{x\to1} \frac{x^3-1}{x-1} = \lim_{x\to1} \frac{(x-1)(x^2+x+1)}{x-1} = \lim_{x\to1} \left(x^2+x+1\right) = 3.$

C. Case $\frac{\infty}{\infty}$: divide numerator and denominator by the highest power in the denominator.

__Example__ (c) $\lim_{x\to+\infty} \dfrac{4x^2+x-1}{x^3+1} = \lim_{x\to+\infty} \dfrac{4/x + 1/x^2 + 1/x^3}{1+1/x^3} = \dfrac{0}{1} = 0.$

(b) $\lim_{x\to+\infty} \dfrac{\sqrt{x^2+x}}{x} = \lim_{x\to+\infty} \dfrac{\sqrt{1+1/x}}{1} = 1.$

D. Functions with roots: cases $\frac{0}{0}$ and $\infty-\infty$: multiply and divide by __conjugate__ root.

__Example__ (a) $\lim_{x\to0} \dfrac{x}{1-\sqrt{1-x}} = \lim_{x\to0} \dfrac{x}{1-\sqrt{1-x}}\cdot\dfrac{1+\sqrt{1-x}}{1+\sqrt{1-x}} = \lim_{x\to0} \dfrac{x(1+\sqrt{1-x})}{1-(1-x)}$

$= \lim_{x\to0} \dfrac{x(1+\sqrt{1-x})}{x} = 2.$

(b) $\lim_{x\to+\infty}\left(\sqrt{x^2+1}-x\right) = \lim_{x\to+\infty}\left(\sqrt{x^2+1}-x\right)\dfrac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x} = \lim_{x\to+\infty}\dfrac{(x^2+1)-x^2}{\sqrt{x^2+1}+x} = 0.$

__CONTINUOUS FUNCTIONS__

A function $f$ is continuous at $x=a$ if

(1) $f(a)$ exists __and__ (2) $\lim_{x\to a} f(x) = f(a)$

__Examples__. (a) $f(x) = \sin(x) + x^3 + 2x$ is continuous at all points.

(b) $f(x) = \sqrt{x+1}$ is continuous at all points of $[-1,+\infty)$

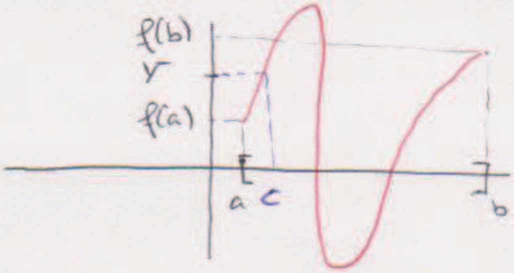
(c) $f(x) = \dfrac{x^2-1}{x-1}$ is continuous on $\mathbb{R}-\{1\}$.

(d) $f(x) = \begin{cases} 2x+3, & x\le 9 \\ 4x^2+7, & x>9 \end{cases}$ is continuous on $\mathbb{R}-\{9\}$.

## IMPORTANT FACTS ABOUT CONTINUOUS FUNCTIONS.

**Intermediate Value Theorem (IVT)**

Suppose $f$ is continuous on the closed interval $[a,b]$ and let $\gamma$ be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists $c$ in $(a,b)$ such that $f(c) = \gamma$.




(Note: there is more than one possibility for c in the picture)

**Application 1: Finding roots of equations.**

Ex. Show that $4x^3 - 6x^2 + 3x - 2$ has a root between 1 and 2.

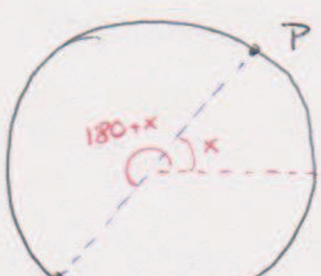
Consider $f(x) = 4x^3 - 6x^2 + 3x - 2$, which is continuous everywhere, so in particular between 1 and 2. Now

$-1 = f(1) \neq f(2) = 12$. By IVT (applied to $\gamma = 0$) there is at least one c in $(1,2)$ such that $f(c) = 0$ (that is, a root)

**Application 2: Atmospheric pressure**

Ex. At any time, there are two antipodal points on Earth that have the same atmospheric pressure (or temperature, for same price)

- Let's find two such points along the equator. Represent points using angle with a horizontal ($x$ in the picture).




Let $A(x)$ be atmospheric pressure at $x$.

Consider the function $F(x) = A(x) - A(\pi + x)$ (continuous) and choose a point $a$ for which $A(a) \neq A(\pi + a)$; otherwise we've found our points. Then

$$F(a) = A(a) - A(\pi + a)$$

$$F(\pi + a) = A(a+\pi) - A(a+2\pi) = A(a+\pi) - A(a) = -F(a)$$

By IVT there exists $c$ in $(a, a+\pi)$ such that $F(c) = 0$, that is, a point $c$ for which $A(c) = A(c+\pi)$.

MAXIMA / MINIMA:

Suppose $f$ is continuous on the closed interval $[a,b]$. Then $f$ attains both a maximum and a minimum on $[a,b]$.