COMPOSING FUNCTIONS

Given two functions f and g, the composition of f and g is the function $f \circ g$ defined by

$$(f \circ g)(x) = f(g(x)).$$

Examples. (a) If $f(x) = \cos(x)$, $g(x) = x^2$, then

$$(f \circ g)(x) = \cos(x^2) \text{ and } (g \circ f)(x) = (\cos(x))^2$$

(b) If $f(x) = x^2$ and $g(x) = x+1$, then

$$(f \circ g)(x) = (x+1)^2 \text{ and } (g \circ f)(x) = x^2 + 1$$

WARNING! As we see in the examples above, in general

$$g \circ f \neq f \circ g$$

Thus be careful about the order in which you compose functions!

Example. If $f(x) = \sin(x)$ and $g(x) = x^3$, decide whether $f \circ g$ is even, odd or neither:

SOL. Have $(f \circ g)(x) = \sin(x^3)$. Now calculate $(f \circ g)(-x)$

$$(f \circ g)(-x) = \sin((-x)^3) = \sin(-x^3) = -\sin(x^3).$$

Since $(f \circ g)(-x) = -(f \circ g)(x)$, the function $f \circ g$ is ODD.

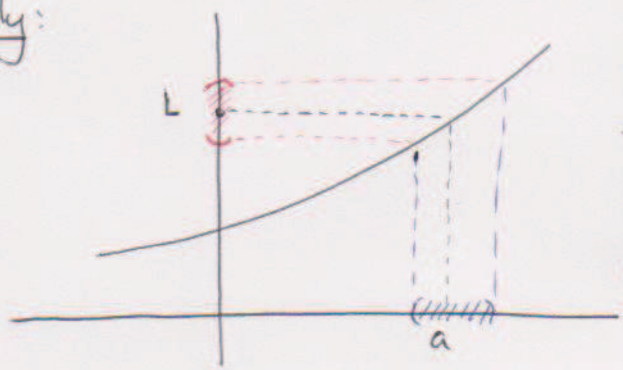
LIMITS.

We write $\boxed{\lim_{x \to a} f(x) = L}$, and say

"the limit of $f(x)$, as x tends to a, equals L"

if we can make all values of $f(x)$ as close to L as we like by taking x to be sufficiently close to a (on either side of a) but not equal to a.

Pictorially:



We can make all values of $f(x)$ to lie in the red region by taking the values of x in the blue region.

OF course, different

Note: $f(a)$ need not be defined for $\lim_{x \to a} f(x)$ to exist!

For instance; if $f(x) = \frac{x-1}{x^2-1}$:

see below

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x-1}{x^2-1} = \lim_{x \to 1} \frac{\cancel{x-1}}{(\cancel{x-1})(x+1)} \overset{(*)}{=} \lim_{x \to 1} \frac{1}{x+1} = \frac{1}{2}$$

I can cancel because $x \neq 1$.

Observe: In $(*)$, I can cancel the $(x-1)$ because, in the definition of limit, ~~x~~ is not allowed to be equal to a (ie, 1 in this case)

But the function $f(x) = \frac{x-1}{x^2-1}$ is NOT THE SAME as $g(x) = \frac{1}{x+1}$

LIMIT LAWS.

- Suppose $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ both exist, and let c be a constant:

(1) $\lim_{x \to a} \left(f(x) + g(x) \right) = \lim_{x \to a} f(x) + \lim_{x \to c} g(x)$

(2) $\lim_{x \to a} \left(c \cdot f(x) \right) = c \cdot \lim_{x \to a} f(x)$

(3) $\lim_{x \to a} \left(f(x) g(x) \right) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

(4) $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ if $\lim_{x \to a} g(x) \neq 0$.

(5) $\lim_{x \to a} x^n = a^n$

(7) $\lim_{x \to a} c = c$.

(6) $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$

Ex. $\lim_{x \to 2} \left(x^2 + 4x + 3 \right) \overset{(1)}{=} \lim_{x \to 2} (x^2) + \lim_{x \to 2} (4x) + \lim_{x \to 2} (3)$

$\overset{(2)}{=} \lim_{x \to 2} (x^2) + 4 \lim_{x \to 2} (x) + \lim_{x \to 2} 3 \overset{(5),(7)}{=} 2^2 + 4 \cdot 2 + 3 = 15$

Example. $\lim_{x \to 3} \dfrac{x+4}{x^2+5} \overset{(4)}{=} \dfrac{\lim_{x \to 3} (x+4)}{\lim_{x \to 3} (x^2+5)} \overset{(5),(7)}{=} \dfrac{7}{14} = \dfrac{1}{2}$.

WHY SOME LIMITS MAY FAIL TO EXIST

A. Because the one-sided limits do not agree.

Write $\boxed{\lim_{x \to a^-} f(x) = L}$ ("the limit of $f(x)$, as f tends to a from the left, equals L")

if we can make all values of $f(x)$ as close as we want to L

by taking x to be sufficiently close to a and to the left of a.

Define $\boxed{\lim_{x \to a^+} f(x) = L}$ in an analogous way.

FACT. $\lim_{x \to a} f(x) = L \iff \left[\lim_{x \to a^-} f(x) = L \text{ and } \lim_{x \to a^+} f(x) = L \right]$

Example. Let $f(x) = \begin{cases} x^2 + 1, & \text{if } x \le 3 \\ x + 8, & \text{if } x > 3. \end{cases}$

Then $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \left(x^2 + 1 \right) = 10$ but

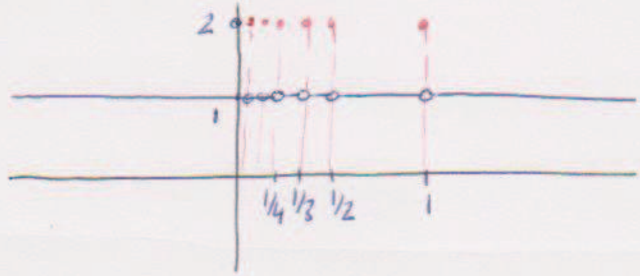
$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} \left(x + 8 \right) = 1$

So $\lim_{x \to 3} f(x)$ does not exist.

Note: all the limit laws hold for calculating one-sided limits

B. WHEN THE FAILURE IS TOTAL (Extra material)

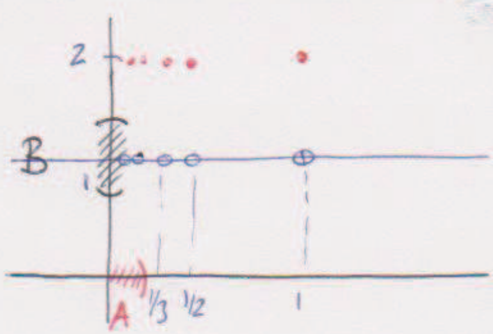
Ex. Consider $f(x) = \begin{cases} 1, & \text{if } x \text{ is not of the form } 1/n,\ n \text{ in } \mathbb{N}. \\ 2, & \text{if } x \text{ is of the form } 1/n,\ n \text{ in } \mathbb{N}. \end{cases}$



We claim that $\lim_{x\to 0} f(x)$ does not exist. In fact, we claim that $\lim_{x\to 0^+} f(x)$ does not exist either (note $\lim_{x\to 0^-} f(x) = 1$)

If this limit existed, it would have to be equal to 1 (WHY?).

But $\lim_{x\to 0^+} f(x) = 1$ means that we could make ALL values of $f(x)$ as close as we want to 1 by choosing x sufficiently close to $x = 0$ on the right of $x = 0$



Consider the black region (B) around $y = 1$ on the picture. If $\lim_{x\to 0^+} f(x) = 1$, we would have to be able to find a red region (A) to the right of $x = 0$ such that, for all x in A, $f(x)$ is in B. But every choice of a region A to the right of $x = 0$ contains ~~points~~ numbers of the form $1/n$ and, for these, the value of $f(x)$ is 2, which is not in B. Thus $\lim f(x)$ cannot be 1 either!