PART 1 : CONTINUOUS FUNCTIONS

[Chapters 1 & 2 from Stewart]

Functions arise whenever one quantity depends on another

Example. Consider the race of the 100m final at the 2009 World Championships in Berlin, and the function $p(t)$, the position of Ussain Bolt on the lane at time t. Also, his velocity $v(t)$

Question: Using the video, calculate $v(4)$ (velocity after 4s.)

Know: $p(4) = 34.0\,m$ $\quad p(4.5) = 41.3\,m$.

Then $$v(4) \approx \frac{p(4.5) - p(4)}{4.5 - 4} \quad \text{(average velocity)}$$

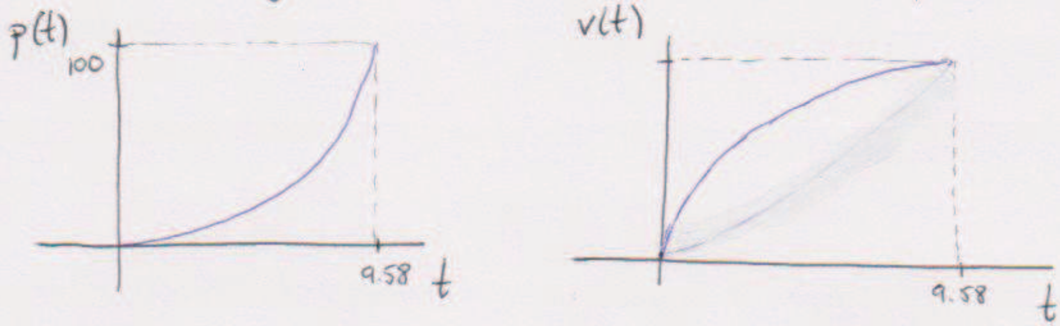
Exact answer: take the LIMIT of average velocities

$$v(4) = \lim_{h \to 0} \frac{p(4+h) - p(4)}{h} \quad \text{(instantaneous velocity)}$$

A FUNCTION is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, of a set E

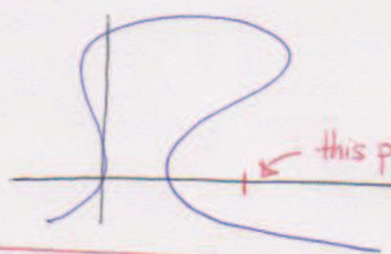
GRAPHS. Represent functions on a graph

Example. What do the graphs of $p(t)$ and $v(t)$ above look like?



MAYBE SOMETHING LIKE THIS?

Example. The curve below is NOT the graph of a function:



this point (for instance) has more than one image!!

Example. Sometimes, functions are given explicitly:

$$f(x) = x^2 - 1 \; ; \; g(x) = \frac{x^2+2}{x-3}.$$

The domain of a function is the set of real numbers for which the function is defined.

Example. (a) $f(x) = x^2 + 1$ has domain $\mathbb{R}$ (the set of all real numbers)

(b) $g(x) = \dfrac{x^2+2}{x-3}$ has domain $\mathbb{R} - \{3\}$ (all real numbers except 3)

(c) $h(x) = \sqrt{x}$ has domain $[0, +\infty)$

(d) $p(x) = \sqrt{x^2-1}$ has domain $(-\infty, -1] \cup [1, +\infty)$

x-intercept: where the graph of f intersects the x-axis
(solve $f(x) = 0$) (MAYBE MANY (or ZERO))

y-intercept: where the graph of f intersects the y-axis,
ie, the point $(0, f(0))$, if defined.

Example. (a) $f(x) = x^2 - 1$.

x-intercept: $f(x) = 0 \Leftrightarrow x^2 - 1 = 0 \Leftrightarrow x = \pm 1$.
So x-intercepts are $(-1,0)$, $(1,0)$.

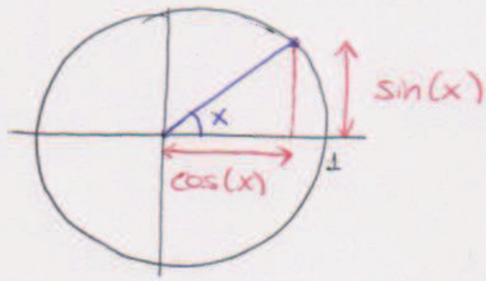
y-intercept: $f(0) = -1$. So, y-intercept is $(0,-1)$

(b) $g(x) = \dfrac{x^2-1}{x}$ has x-intercepts $(-1,0)$ and $(1,0)$ but no y-intercepts

(c) $h(x) = x^2 + 1$ has no x-intercepts, and y-intercept $(0,1)$

SINE AND COSINE

Draw a circle of radius 1. The sine and cosine of an angle x (measured in radians; see below) are depicted in the figure:



From the picture, we can see that:

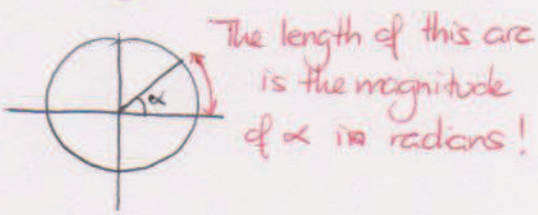
(a) $-1 \leq \sin(x) \leq 1$ and $-1 \leq \cos(x) \leq 1$.

(b) $\sin(-x) = -\sin(x)$ and $\cos(-x) = -\cos(x)$

(c) $\sin(x) = \sin(x + 2\pi)$ and $\cos(x) = \cos(x + 2\pi)$.

(that is, sin and cos are periodic of period 2π)

What is a radian? The magnitude in radians of an angle is the length of the circular arc it determines on the circle of radius 1.



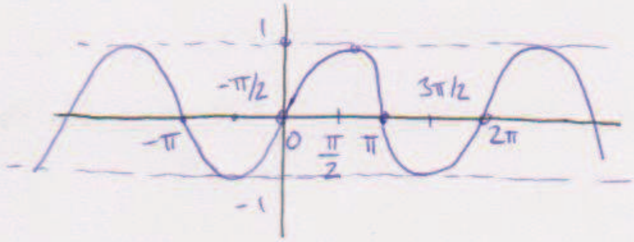
So: $90^\circ = \frac{\pi}{2}$ radians

$180^\circ = \pi$ radians

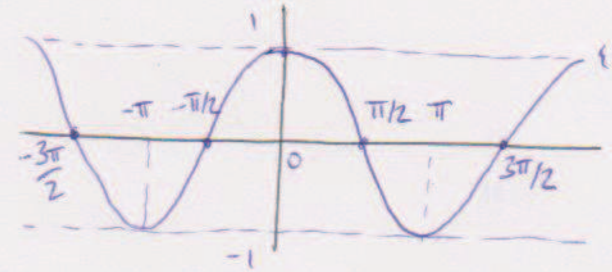
$360^\circ = 2\pi$ radians, etc...

SET YOUR CALCULATOR IN RADIANS NOW!

The graphs of sine and cos are:



sin(x)



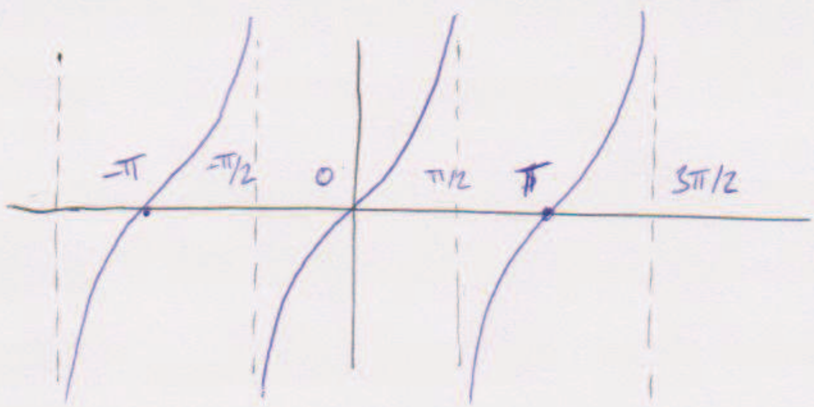
cos(x).

TANGENT.

The tangent function is $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$.

Observe that the domain of tan is $\mathbb{R} - \underbrace{\{\text{odd integer multiples of } \tfrac{\pi}{2}\}}_{\text{values at which } \cos(x) \text{ vanishes.}}$

The graph of tan is



APPLICATIONS. In signal processing, one decomposes a signal as a "sum" of sines and cosines (its simpler components). This is called Fourier Analysis.

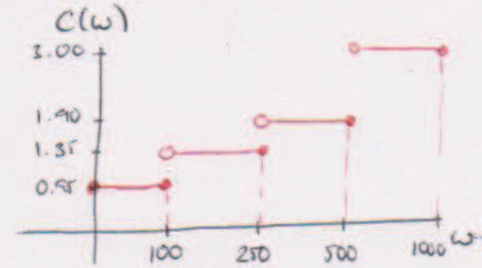
frequency spectrum

FUNCTIONS DEFINED IN PIECES.

Some functions are naturally defined in pieces:

Example. The cost $C(w)$ of sending an envelope to Ireland, in terms of weight w (from An Post website) is (in euros):

$$C(w) = \begin{cases} 0.95, & \text{if } 0 \leq w \leq 100\text{g} \\ 1.35, & \text{if } 100 < w \leq 250 \\ 1.90, & \text{if } 250 < w \leq 500 \\ 3.00, & \text{if } 500 < w \leq 1000 \end{cases}$$



Example. The absolute value of x is

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

(e.g. $|7| = 7$, $|-5| = -(-5) = 5$)

SYMMETRIES.

- A function f is said to be **even** if $f(x) = f(-x)$ for all x
- A function f is said to be **odd** if $f(-x) = -f(x)$ for all x

Example. (a) $f(x) = \cos(x)$ is even since $\cos(-x) = \cos(x)$ for all x.

(b) $f(x) = x^{2n}$ is also even, since $(-x)^{2n} = x^{2n}$ "

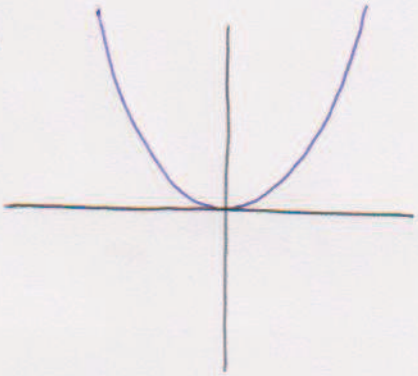
(c) $f(x) = \sin(x)$ is odd, since $\sin(-x) = -\sin(x)$ for all x

(d) $f(x) = x^{2n+1}$ is odd.

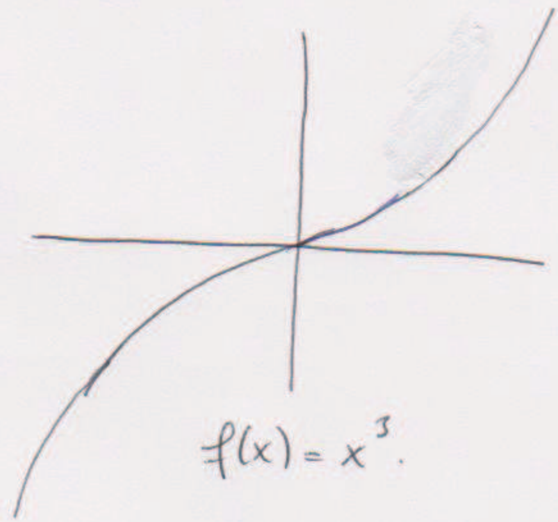
The graph of an even function is symmetric about the y-axis.

The graph of an odd function is symmetric about the origin.

Example.



$f(x) = x^2$



$f(x) = x^3$.