ANTIDERIVATIVES.

Example. Observations suggest that the rate of growth of a population of zebra mussels is exponential in time (in days). Suppose $\boxed{\frac{dN}{dt} = e^{2t}}$

where $N(t)$ is the number of zebra mussels at time t. If there are 150 mussels at time $t=0$, how many will there be after 10 da

Sol. If $\frac{dN}{dt} = e^{2t}$ then $N(t) = \frac{1}{2}e^{2t} + c$ (CHECK!)

(c is a constant). Now $N(0) = 150$ so

$150 = \frac{1}{2}e^{0} + c$, and thus $c = 149.5$. Hence

$$\boxed{N(t) = \frac{1}{2}e^{2t} + 149.5}$$

At $t = 10$, $N(10) = \frac{1}{2}e^{20} + 149.5 \simeq 2.4 \times 10^{8}$ mussels.

The function $N(t) = \frac{1}{2}e^{2t} + 149.5$ is an antiderivative of the function $f(t) = e^{2t}$, since $N'(t) = f(t)$.

A function F is called an antiderivative of f if

$$F' = f.$$

If F is an antiderivative of f then all antiderivatives of f have the form:

$$F(x) + c, \quad c \text{ a constant.}$$

Examples. (a) If $f(x) = x$ then $F(x) = \dfrac{x^2}{2} + c$.

(b) If $f(x) = x^2$ then $F(x) = \dfrac{x^3}{3} + c$

(c) In general, if $f(x) = x^n$ then $F(x) = \dfrac{x^{n+1}}{n+1} + c$. $(n \neq -1)$.

(d) If $f(x) = \cos(x)$ then $F(x) = \sin(x) + c$.

(e) If $f(x) = \sin(x)$ then $F(x) = -\cos(x) + c$.

(f) If $f(x) = e^x$ then $F(x) = e^x + c$.

(g) If $f(x) = \dfrac{1}{x}$ then $F(x) = \ln(x) + c$.
$(x > 0)$

(h) If F is an antiderivative of f and G is an antiderivative of g then $F + G$ is an antiderivative of $f + g$.

Exercise. Find an antiderivative of

$$f(x) = e^x + x^2 + 3x^4 + \cos(x).$$

$$F(x) = e^x + \frac{x^2}{2} + 3 \cdot \frac{x^4}{4} + \sin(x).$$

Indeed, check that $F'(x) = f(x)$.

APPLICATIONS OF EXPONENTIAL & LOGARITHMIC FUNCTIONS.

A. POPULATION MODELS

Example (generic). A population of bacteria has rate of growth equal to twice the size of the population, based on experimental data. Let $p(t)$ be the size of the population at time t (t measured in seconds). Suppose that there are 2000 bacteria at $t=0$.

(i) What will the population be after 4 seconds?

(ii) How long will it take for the population to reach 20000 individuals?

Sol. We are told $\boxed{p'(t) = 2p(t).}$ $(*)$

By inspection we get $p(t) = e^{2t}$ is ONE solution to equation $(*)$, since $\frac{d}{dt}(e^{2t}) = 2e^{2t}$.

The general solution of $(*)$

$$\boxed{p(t) = Ce^{2t}, \text{ for } C \text{ a constant}}$$

(this is the exponential model for population growth).

Now, $p(0) = 2000$, so

$$2000 = Ce^{2\cdot 0} = C$$

Thus $\boxed{p(t) = 2000e^{2t}}$ is the solution to our problem.

(i) $p(4) = 2000e^{8} \approx 5.96 \cdot 10^{6}$ individuals.

(ii) Want to find the value of t at which $p(t) = 20000$.

$$20000 = 2000e^{2t} \quad \text{so}$$

$$e^{2t} = 10 \quad \underline{\text{or}} \quad t = \frac{\ln 10}{2} \simeq 1.151 \text{ seconds}.$$

<u>Example</u> (Estimating the proportionality constant).

The <u>World's population</u> was 2560 millions in 1950 and 3040 m. in 1960. Assume growth rate of population is proportional to its size.

(a) Estimate the population in ~~[illegible]~~. 1990 and 2010

(b) Predict " " " 2040.

<u>Sol</u>. ~~We are told~~ Let $p(t)$ be population (in millions). at time t, where we measure t in years. Set $t = 0$ at 1950. As before,

$$\boxed{p'(t) = kp(t)} \quad (*)$$

(k a constant to be determined).

Therefore, as earlier,

$$\boxed{p(t) = Ce^{kt}, \quad C \text{ a constant}}$$

is <u>the</u> <u>general</u> <u>solution</u> of $(*)$.

Now, $p(0) = 2560$ so $C = 2560$. Thus

$$\boxed{p(t) = 2560\, e^{kt}.}$$

Also, $p(10) = 3040$ so $3040 = 2560\, e^{k10}$.

Thus $\boxed{k = \frac{1}{10} \ln\left(\frac{3040}{2560}\right) \simeq 0.017185.}$

and $\boxed{p(t) = 2560\, e^{0.017185t}}$

(a) $p(40) = 2560\, e^{0.017185 \cdot 40} \approx 5090$ millions.

(actual figure ≈ 5278 millions (US Census Bureau))

• $p(60) = 2560\, e^{0.017185 \cdot 60} \approx 7178$ millions

(actual figure ≈ 6848 millions)

(b) $p(90) \approx 12021$ millions (actual prediction 8850 millions).

So our exponential model for population growth is not very realistic in the long run; in particular it predicts that

$$\lim_{t \to \infty} p(t) = +\infty.$$

However, it produces reasonable answers for "small" values of t.

More realistic assumptions for a population model could be :

(i) $\frac{dp(t)}{dt} \approx kp(t)$ (for small $p(t)$)

(that is, initially the growth rate is proportional to size)

(ii) $\frac{dp(t)}{dt} < 0$ if $p(t) > K$ where K is a number that represents the carrying capacity of the system (i.e, the maximum number of individuals the system can possibly take).

An equation satisfying (i) and (ii) is

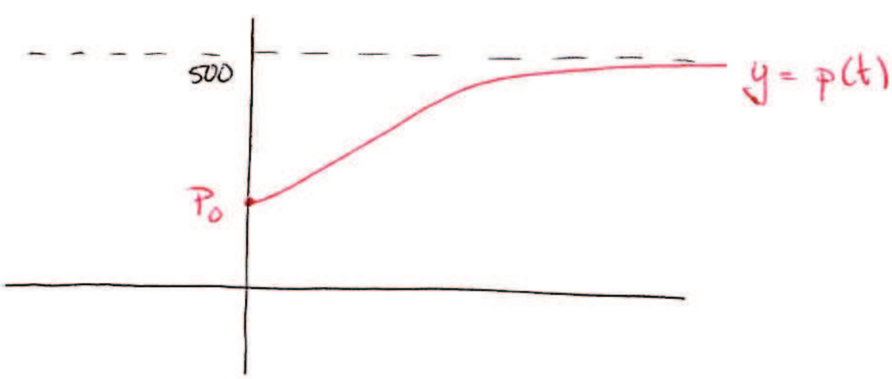
$$\boxed{\frac{dp(t)}{dt} = kp(t)\left(1 - \frac{p(t)}{K}\right)} \quad (*)$$

The solution to $(*)$, subject to the condition that $p(0) = P_0$, is (CHECK!)

$$\boxed{p(t) = \frac{K}{1 + Ae^{-kt}}, \quad \text{where} \quad A = \frac{K - P_0}{P_0}.}$$

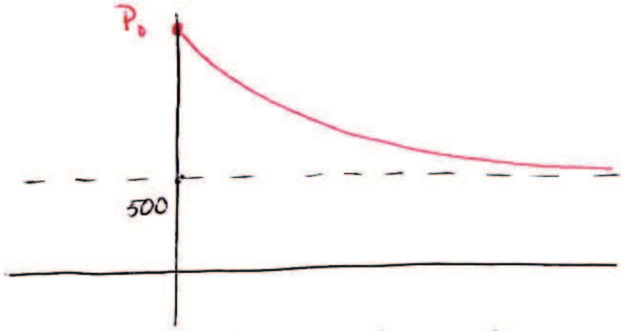
* Let's set, for simplicity, that $k = 1$ and $K = 500$.

Case 1. Our initial population P_0 is less than 500.



(Increase and stabilize at 500).

Case 2. Our initial population P_0 is more than 500.



(Decrease and stabilize at 500)

Case 3. If $P_0 = 500$ then $p(t) = 500$ for all t.

B. RADIOACTIVE DECAY.

Radioactive materials decay when they emit radiation. Let $m(t)$ be the mass at time t. Let's assume that the rate of decay is proportional to mass, so

$$m'(t) = k\,m(t)$$

but now $k < 0$!

Example. Radium-226 has a half-life of 1590 years (half-life: time required for half of any given quantity of radioactive material to decay).

(a) Consider a mass of 100 mg of Ra-226. Find a formula for the mass of the sample that remains after t years.

(b) Find the mass after 1000 years, correct to the nearest milligram.

(c) When will the mass be reduced to 30g?

Let $m(t)$ be the mass (in milligrams) after t years.

Then
$$\boxed{\begin{aligned} m'(t) &= km(t) \\ m(0) &= 100 \end{aligned}}$$

so $\boxed{m(t) = 100e^{kt}}$ Now half-life is 1590 years,

so we can determine k:

$$50 = 100e^{k \cdot 1590}, \quad \text{so } k = -\frac{\ln 2}{1590}. \quad \text{Thus}$$

(a) $\boxed{m(t) = 100e^{-\frac{\ln 2}{1590} \cdot t}}$

(b) $m(1000) = 100e^{(-\ln 2/1590) \cdot 1000} \approx 65 \text{ mg}.$

(c) $30 = 100e^{(-\ln 2/1590)t}$, so $t \approx 2762$ years.