A bit more generally:

Cauchy's Mean Value Theorem. Suppose f, g are continuous on $[a,b]$ and differentiable on (a,b), and that $g'(x) \neq 0$ for all x in (a,b).
Then there exists a number c in (a,b) such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

Proof. Let $h(x) = f(x) - f(a) - \frac{f(b)-f(a)}{g(b)-g(a)}\left[g(x)-g(a)\right]$

[Check: h satisfies the hypotheses $\overbrace{\text{of}}^{}_{(1),(2)}$ Rolle's Theorem.]

Now, $h(a) = 0$ and $h(b) = 0$ as well. So there is a number c in (a,b) such that $h'(c) = 0$. Thus

$$0 = f'(c) - \frac{f(b)-f(a)}{g(b)-g(a)}\, g'(c)$$

Rearranging we obtain the desired result ■

A consequence of Cauchy's MVT is L'Hôpital's Rule for calculating limits:

L'Hôpital's Rule: Suppose f, g are differentiable and $g'(x) \neq 0$ near a (maybe except at a). Suppose

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

has the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Then $\displaystyle\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$.

Note: The same applies if $x \to a^+$, $x \to a^-$, $x \longrightarrow +\infty$, $x \longrightarrow -\infty$

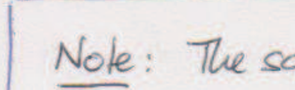
Proof of L'Hôpital's Rule. DO IT FOR FORM $\frac{0}{0}$, $x \longrightarrow a^+$

Let $F(x) = \begin{cases} f(x), & \text{if } x \neq a \\ 0, & \text{if } x = a \end{cases}$ $\quad G(x) = \begin{cases} g(x), & \text{if } x \neq a \\ 0, & \text{if } x = a. \end{cases}$

Then F is continuous since $\displaystyle\lim_{x \to a} F(x) = \lim_{x \to a} f(x) = 0 = F(0)$.

Similarly, G is continuous.

Moreover, F, G satisfy the hypotheses of Cauchy's MVT (Why?)



Let $x > a$. By Cauchy's MVT there is y in (a, x) such that

$$\frac{F'(y)}{G'(y)} = \frac{F(x) - F(a)}{G(x) - G(a)} = \frac{F(x)}{G(x)}.$$

Moreover, $y \to a^+$ if $x \to a^+$. Thus

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{F(x)}{G(x)} = \lim_{x \to a^+} \frac{F'(x)}{G'(x)} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)}. \blacksquare$$

Example. (a) $\displaystyle\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{\cos(x)}{1} = \cos(0) = 1.$

(b) $\displaystyle\lim_{x \to 0} \frac{\tan(x)}{x} = \lim_{x \to 0} \frac{\frac{1}{\cos^2 x}}{1} = \lim_{x \to 0} \frac{1}{\cos^2(x)} = 1.$

PART 2

"Rates of Change and Optimization" (Ch. 2, 3 from Stewart)

The derivative of the function f at the point $x = a$ is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad \text{(if limit exists)}$$

Equivalently, $$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad \text{(write } x = a+h\text{)}$$

The function that assigns to every x the value $f'(x)$ is called the derivative function of f. We write f', or $\frac{df}{dt}$.

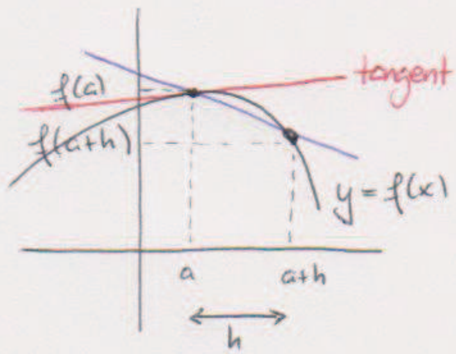
- Physical interpretation of the derivative:

  If $f(x)$ is a quantity that depends on x, then $f'(x)$ is the instantaneous rate of change of f at x.

In particular,

$$\text{if } f \text{ is } \begin{Bmatrix} \text{position} \\ \text{velocity} \\ \text{cost} \\ \text{population} \end{Bmatrix} \text{ then } f' \text{ is } \begin{Bmatrix} \text{velocity} \\ \text{acceleration} \\ \text{marginal cost} \\ \text{growth rate.} \end{Bmatrix}$$

Interpetation in terms of tangent line:

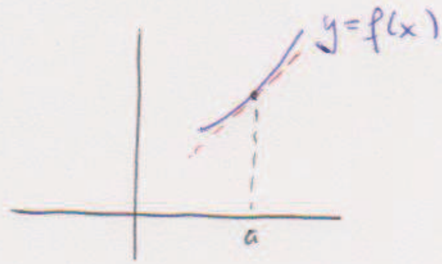


- $\dfrac{f(a+h)-f(a)}{h}$ is the slope of the line through $(a, f(a))$ and $(a+h, f(a+h))$
- Thus $f'(a)$ is the slope of the line tangent to $y = f(x)$ at $(a, f(a))$
- So the equation of the tangent line is

$$y - f(a) = f'(a)(x-a)$$

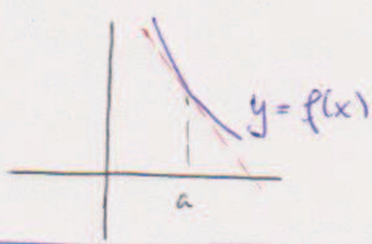
WHAT f' SAYS ABOUT THE GRAPH OF f.

1A. If $f'(x) > 0$ near a then f increases near a:



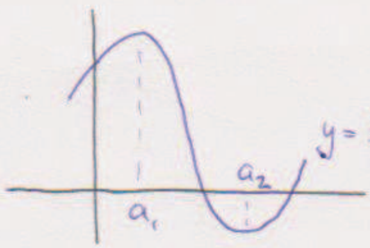
- f increases near a: positive derivative, so positive slope of the tangent line.

1B. If $f'(x) < 0$ near a then f decreases near a



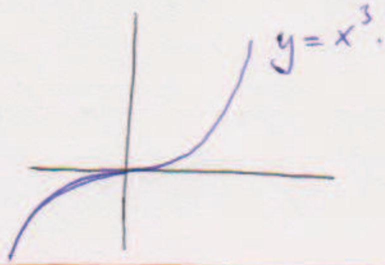
- f decreases near a: negative derivative, so negative slope of tangent line.

2. If f has a <u>LOCAL MAXIMUM</u> or a <u>LOCAL MINIMUM</u> at $x = a$, then $f'(a) = 0$.



- f has a LOCAL MAX at a_1 and a local MIN at a_2: horizontal tangent, i.e., zero derivative.

WARNING. There are functions f that satisfy $f'(a) = 0$ for some a, but they have neither a maximum nor a minimum at $x = a$. One example is $f(x) = x^3$ at $x = 0$.



RULES OF DIFFERENTIATION.

(a) <u>Power rule</u>: $\frac{d}{dx}(x^n) = n x^{n-1}$, where n is any number.

Ex. $\frac{d}{dx}(x^3) = 3x^2$; $\frac{d}{dx}(x^{3/2}) = \frac{3}{2} x^{1/2}$; $\frac{d}{dx}(x^{\sqrt{2}}) = \sqrt{2} x^{\sqrt{2}-1}$

(b) If c is any number and f is differentiable:

$$\frac{d}{dx}(c f(x)) = c \frac{d}{dx} f(x)$$

Ex. $\frac{d}{dx}(3x^4) = 3 \frac{d}{dx}(x^4) = 12 x^3$.

(c) SUM RULE. $\frac{d}{dx}\left(f(x)+g(x)\right) = \frac{d}{dx} f(x) + \frac{d}{dx} g(x).$

Ex. $\frac{d}{dx}\left(3x^2 + 4x^3\right) = 6x + 12x^2.$

(d) PRODUCT RULE $\frac{d}{dx}\left(f(x)g(x)\right) = \left[\frac{d}{dx} f(x)\right] \cdot g(x) + f(x) \cdot \left[\frac{d}{dx} g(x)\right]$

Ex. $\frac{d}{dx}\left(x(x^2+1)\right) = 1 \cdot (x^2+1) + x \cdot 2x = 3x^2 + 1.$

(e) QUOTIENT RULE. $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x) \cdot g(x) - f(x) g'(x)}{[g(x)]^2}$

Ex. $\frac{d}{dx}\left[\frac{x}{x^2+1}\right] = \frac{1 \cdot (x^2+1) - x \cdot 2x}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}.$

(f) TRIGONOMETRIC FUNCTIONS

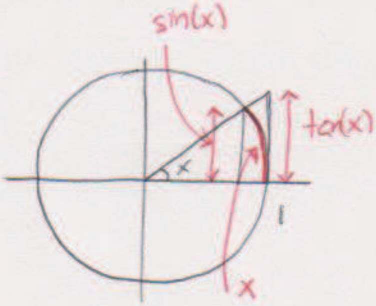
$\frac{d}{dx}\left(\sin(x)\right) = \cos x$; $\frac{d}{dx}\left(\cos(x)\right) = -\sin(x).$

• WHY IS COS THE DERIVATIVE OF SIN ??

Let's convice ourselves of this : let's prove that if $f(x) = \sin(x)$ then $f'(0) = \cos(0) = 1.$

$$f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{\sin(x)-\sin(0)}{x} = \lim_{x\to 0} \frac{\sin(x)}{x}$$

So we have to convince ourselves that $\lim_{x\to 0} \frac{\sin x}{x} = 1$.



From the picture: $\sin(x) < x < \tan(x)$

Divide by $\sin(x)$: $1 < \frac{x}{\sin(x)} < \frac{1}{\cos(x)}$, or

$$\cos(x) < \frac{\sin(x)}{x} < 1.$$

Since $\lim_{x\to 0} \cos(x) = \cos(0) = 1$ (because $\cos(x)$ is continuous)

and $\lim_{x\to 0} 1 = 1$, the Squeeze Theorem says $\boxed{\lim_{x\to 0} \frac{\sin(x)}{x} = 1.}$

(g) THE CHAIN RULE. Let $F(x) = (f\circ g)(x) = f(g(x))$. Then

$$F'(x) = f'(g(x)) \cdot g'(x).$$

Ex. $\frac{d}{dx}\left(\sin(x^4)\right) = \cos(x^4)\cdot 4x^3$

$\frac{d}{dx}\left((x^2+2)^7\right) = 7(x^2+2)^6 \cdot 2x.$

FACT. If f is differentiable at $x=a$, then it is continuous at $x=a$.

Proof. Want to show $\lim_{x\to a} f(x) = f(a)$ or, equiv., $\lim_{x\to a}\left[f(x)-f(a)\right] = 0$.

$$\lim_{x\to a}\left[f(x)-f(a)\right] = \lim_{x\to a}\left[(f(x)-f(a))\frac{x-a}{x-a}\right] =$$

$$= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \cdot \lim_{x\to a}(x-a) = f'(a)\cdot 0 = 0, \text{ as desired} \blacksquare$$

since $f'(a)$ exists

THE CONVERSE IS NOT TRUE: Check that if $f(x) = |x|$ then $f'(0)$ does not exist!!

HOW TO FIND MAXIMA / MINIMA

[Recall 1A, 1B, 2 from pages 2 & 3]

(a) Find all critical points of f, that is, those numbers x such that $f'(x) = 0$

(there may be no critical points, though!)

(b) Suppose $x = c$ is a critical point (so $y = f(x)$ has a horizontal tangent at $(c, f(c))$)

(b1) If $f'(x) < 0$ before $x = c$ and $f'(x) > 0$ after $x = c$ (near $x = c$) then $x = c$ is a LOCAL MINIMUM.

(b2) If $f'(x) > 0$ before $x = c$ and $f'(x) < 0$ after $x = c$ (near $x = c$) then $x = c$ is a LOCAL MAXIMUM

(b3) If $f'(x)$ does not change sign, near $x = c$, then $x = c$ is NEITHER A MAXIMUM NOR A MININUM.

Ex. Let $f(x) = 3x - x^3$.

(a) Find all critical points of f. classifying each as a maximum / minimum / neither.

(b) Find all intervals on which f increases / decreases.

$f'(x) = 3 - 3x^2$. Critical points are solutions of $f'(x) = 0$, so $x = +1$, $x = -1$.

$f' < 0$	-1	$f' > 0$	1	$f' < 0$
(since $f'(-2) = -9 < 0$, for instance)		(since $f'(0) = 3 > 0$, for instance)		(since $f'(2) = 3 - 12 = -9 < 0$, for instance)

Thus (by above):

(a) (i) $x = -1$ is a LOCAL MIN, by (b1)

(ii) $x = 1$ is a LOCAL MAX, by (b2)

(b) f increases on $(-1,1)$ and

f decreases on $(-\infty, -1)$ and $(1, +\infty)$

<u>Example</u>. (Solved in class) Two magnets of variable intensity are placed at the extremes of a wooden rod. An iron particle is placed on the rod. We know that the position of the particle is described by the function:

$$s(t) = t^3 - 6t^2 + 9t$$

(a) Find its velocity after 4 seconds.

(b) When is the particle at rest?

(c) When is the particle moving forward? (ie, in the positive direction)

(d) When is the particle speeding up? Slowing down?

<u>ROLLE'S THEOREM</u>

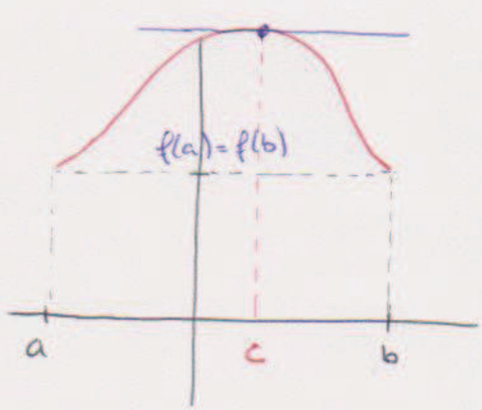
Suppose that f satisfies:

(1) f is continuous on $[a,b]$

(2) f is differentiable on (a,b)

(3) $f(a) = f(b)$

Then there is at least one number c in (a,b) such that $f'(c) = 0$.



Example. Show that the equation $x^5 + 2x + 27$ has a unique real root.

Let $f(x) = x^5 + 2x + 27$, which is differentiable (and thus continuous) everywhere.

Since $f(-10) < 0$ and $f(10) > 0$ (say), we know that there is at least one number a [in $(-10, 10)$] for which $f(a) = 0$. (IVT)

So there is at least one solution. Suppose there were two solutions $a \neq b$; so $f(a) = f(b) = 0$. By Rolle's Theorem, there is then a number c in (a,b) with $f'(c) = 0$. But

$$f'(x) = 5x^4 + 2 \text{ is always positive!}$$

Thus our assumption that there were two solutions is not true, and so there is only one solution.

A BIT MORE GENERALLY:

MEAN VALUE THEOREM (MVT)

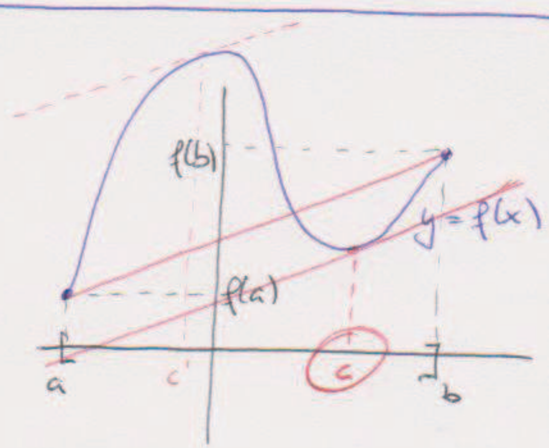
Suppose that f satisfies:

(1) f continuous on $[a,b]$

(2) f differentiable on (a,b)

Then there is a number c in (a,b) such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$



CONSEQUENCES OF MVT

① If $f'(x) = 0$ for all x in (a,b) then f is constant in (a,b).

Proof. Take any x_1, x_2 in (a,b) : let's prove that $f(x_1) = f(x_2)$.

By MVT, there is c in (x_1, x_2) with

$$f(x_2) - f(x_1) = f'(c) \cdot (x_2 - x_1)$$

But $f'(c) = 0$ so $f(x_2) = f(x_1)$ ■

② If $f'(x) = g'(x)$ for all x in (a,b) then

$$f(x) = g(x) + c, \text{ for some constant } c$$

Proof. Define a function F by $F(x) = f(x) - g(x)$.

Then $F'(x) = f'(x) - g'(x) = 0$. By ①, $F(x) = c$ (c constant) for all x in (a,b). In other words, $f(x) = g(x) + c$