Summary:

- we've studied limits

$$\lim_{x \to c} f(x)$$

and used them to study continuity, and briefly used them to study the speed of a stone.

- we've used

$$\lim_{x \to \infty} f(x), \quad \lim_{x \to -\infty} f(x)$$

and $\lim_{x \to c} f(x)$

to find horizontal & vertical asymptotes when graph sketching.

Topic II:

Rates of change

Differentiation

Given a function $f(x)$ we define the derivative to be the function $f'(x)$ defined as

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

Example Find the derivative $f'(x)$ for $f(x) = x^2$.

Soln

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2hx + h^2 - \cancel{x^2}}{h}$$

$$= \lim_{h \to 0} 2x + h$$

$$= 2x$$

Derivatives of some basic functions

For $y = f(x)$ we often write $\frac{dy}{dx}$ instead of $f'(x)$.

- $\frac{d}{dx} x^n = n x^{n-1}$ for any number n.

- $\frac{d}{dx} \sin(x) = \cos(x)$

- $\frac{d}{dx} \cos(x) = -\sin(x)$

- $e = 2.71\ldots$

$$\frac{d}{dx} e^{x} = e^{x}$$

Rules of differentiation

$$\boxed{\bullet\ \frac{d}{dx}\left(f(x) + g(x)\right) = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)}$$

Sum Rule

Example

$$\frac{d}{dx}\left(x^{\frac{3}{2}} + \sin(x)\right)$$

$$= \frac{d}{dx} x^{\frac{3}{2}} + \frac{d}{dx} \sin(x)$$

$$= \frac{3}{2} x^{\frac{1}{2}} + \cos(x)$$

$$\boxed{\bullet\ \frac{d}{dx}\left(k f(x)\right) = k \frac{d}{dx} f(x)}$$

Scalar product rule

Example

$$\frac{d}{dx}(3e^x) = 3\frac{d}{dx}e^x = 3e^x.$$

$$\boxed{\begin{aligned} &\bullet\ \frac{d}{dx}\left(f(x)\,g(x)\right) \\ &= \left[\frac{d}{dx}f(x)\right]g(x) + f(x)\left[\frac{d}{dx}g(x)\right] \end{aligned}}$$

Product Rule

Example

$$\frac{d}{dx}\left(x^2\sin(x)\right)$$

$$= \left[\frac{d}{dx}x^2\right]\sin(x) + x^2\left[\frac{d}{dx}\sin(x)\right]$$

$$= 2x\sin(x) + x^2\cos(x)$$

Example

$$y = (x^2+1)(x^3+2)$$

$$\frac{dy}{dx} = \left[\frac{d}{dx}(x^2+1)\right](x^3+2) + (x^2+1)\left[\frac{d}{dx}(x^3+2)\right]$$

$$= 2x(x^3+2) + (x^2+1)(3x^2)$$

$$= 5x^4 + 3x^2 + 4x$$

Chain Rule

Given functions

$f(x)$ and $g(x)$

we can consider the composite function

$$y = g(f(x))$$

$$\boxed{\bullet\ \frac{dy}{dx} = g'(f(x)) \cdot f'(x)}$$

Chain Rule

Example

$$y = \sin(x^2)$$

$g(x) = \sin(x)$ $\quad$ $f(x) = x^2$

$$\frac{dy}{dx} = \cos(x^2) \cdot 2x$$

$$= 2x \cos(x^2)$$

Example

$$y = (x^2 - x + 1)^7$$

$$\frac{dy}{dx} = 7(x^2 - x + 1)^6 (2x - 1)$$

Example

$$y = \sqrt{x^2 + 1}$$

$$y = (x^2 + 1)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{\cancel{2}}(x^2 + 1)^{-\frac{1}{2}} \cancel{2}x$$

$$= \frac{x}{(x^2+1)^{\frac{1}{2}}} = \frac{x}{\sqrt{x^2+1}}$$

Quotient Rule

$$y = \frac{f(x)}{g(x)}$$

So

$$y = f(x)\left[g(x)\right]^{-1}$$

$$\frac{dy}{dx} = f'(x)\left[g(x)\right]^{-1} - \left[g(x)\right]^{-2} g'(x)\, f(x)$$

$$= \frac{f'(x)}{g(x)} - \frac{g'(x)\, f(x)}{g(x)^2}$$

$$= \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2}$$

QUOTIENT RULE

<u>Exercise:</u> For

$$y = \frac{x^2 + x - 2}{x^3 + 6}$$

find $\dfrac{dy}{dx}$.