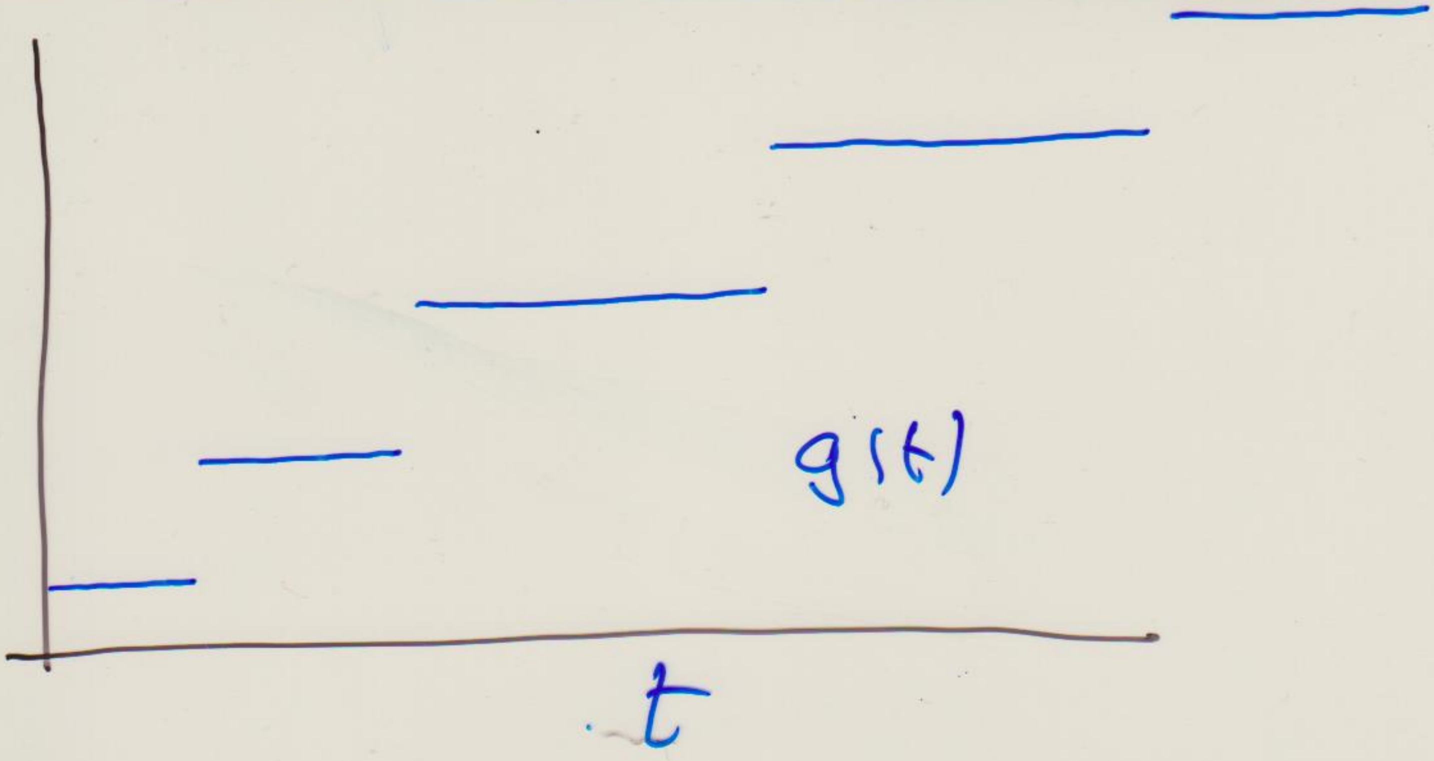
Continuity

h travel to Dublin Airport.

f(t) = distance from Galway
t minutes after
leaving

g(t) = price of car parking
t minutes after
entering car park.



continuous means no jumps in graph.

Better definition:

A function $f(t)$ is continuous if a small change in the input can yield only a small change in the output.

In above example, $f(t)$ is continuous, $g(t)$ is not continuous.

An even better definition.

We say that $g(t)$ is continuous at a point $t=a$ if:

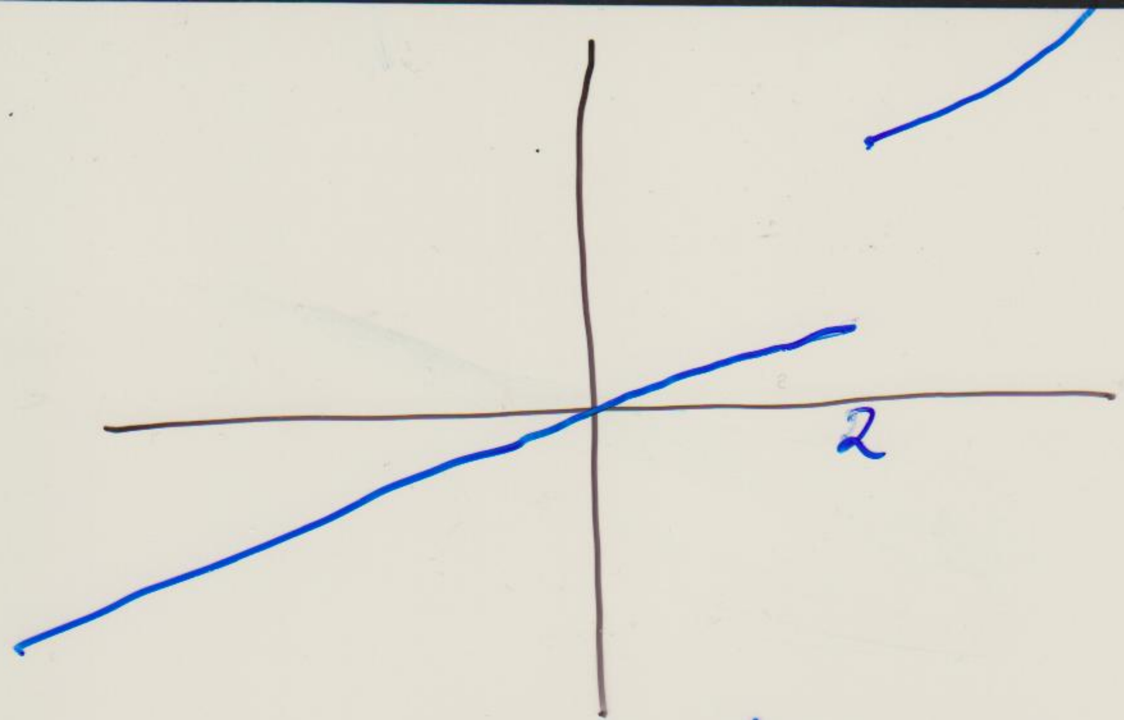
i) $g(a)$ is defined

ii) $\lim_{t \to a} g(t) = g(a)$

Example Determine the constant k such that

$$f(x) = \begin{cases} x^3 & \text{for } x \geq 2 \\ kx & \text{for } x < 2 \end{cases}$$

is continuous at all points.



For continuity at $x = 2$
we need

$$\lim_{x \to 2} f(x) = f(2) = 8$$

i.e. we need

$$\lim_{x \to 2-} f(x) = 8 = \lim_{x \to 2+} f(x)$$

$$\lim_{x \to 2+} f(x) = 8$$

$$\lim_{x \to 2-} f(x) = 2k$$

For continuity need $8 = 2k$

or $k = 4$.

Challenge:
For what value of k is

$$f(x) = \begin{cases} \dfrac{\sin(x)}{x} & , x \neq 0 \\ k & , x = 0 \end{cases}$$

continuous.

Intermediate Value Theorem

Suppose

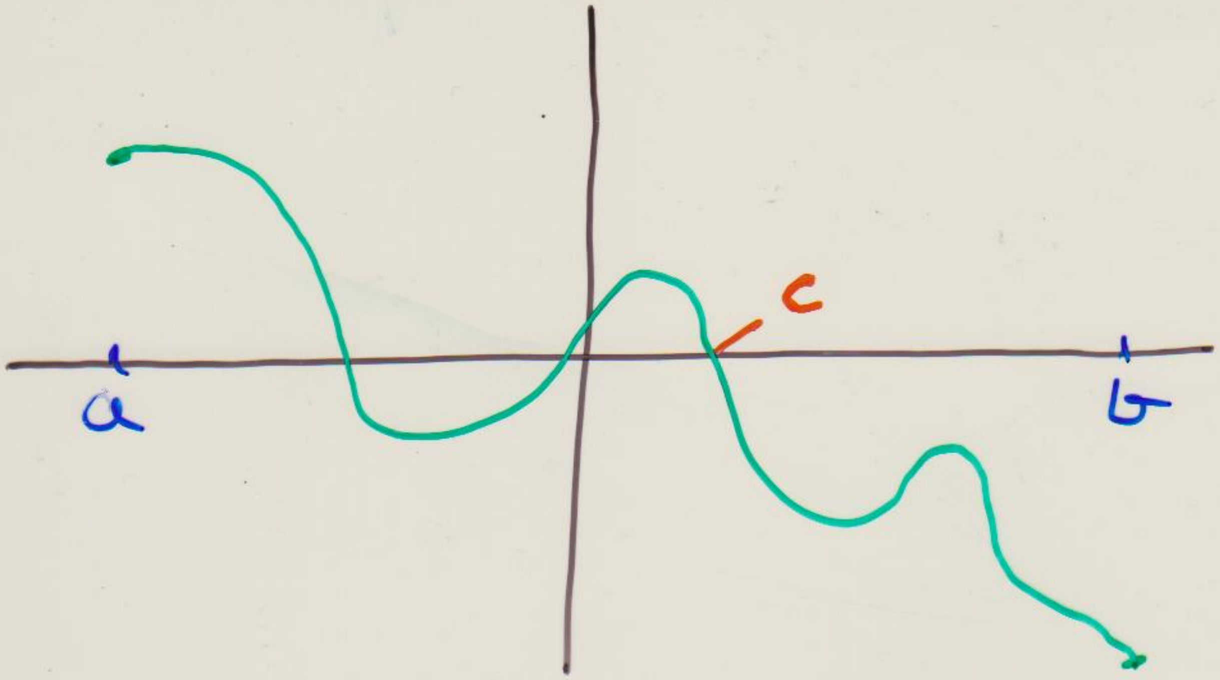
$$y = f(x)$$

is continuous at all points x in a range $a \le x \le b$.

Suppose also that $f(a)f(b) \le 0$.

Then there exists at least one value c in the range $a \le c \le b$ such that

$f(c) = 0$.



$f(a)f(b) \leq 0$

<u>Example</u> show that

$$x^3 - x - 1 = 0$$

has a solution in the range $1 \leq x \leq 2$.

<u>Sol^n</u>

$f(x) = x^3 - x - 1$.

"clearly" $f(x)$ is continuous.

$a = 1 \qquad b = 2$

$f(1) < 0 \qquad f(2) > 0$

so $f(1)f(2) < 0$

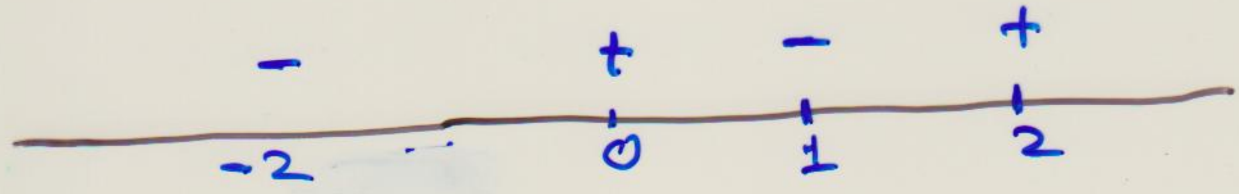
The IVT thus implies that $f(c) = 0$ for some $1 \le c \le 2$.

QED

<u>Example</u> Show that $x^3 - 4x + 1 = 0$ has three real solutions.

<u>Sol^n</u>

$$f(x) = x^3 - 4x + 1$$



$f(0) > 0$

$f(1) < 0$

$f(-3) < 0$

$f(2) > 0$

IVT says there exist $c_1 \in [-2, 0]$, $c_2 \in [0, 1]$, $c_3 \in [1, 2]$ with $f(c_1) = 0$, $f(c_2) = 0$, $f(c_3) = 0$.