Proposition

Suppose

$$\lim_{x \to a} f(x) \quad \text{and} \quad \lim_{x \to a} g(x)$$

both exist. Then

(i) $$\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to b} g(x)$$

(ii) $$\lim_{x \to a} c.f(x) = c. \lim_{x \to a} f(x).$$

(iii) $$\lim_{x \to a} (f(x)\, g(x)) = \left(\lim_{x \to a} f(x)\right)\left(\lim_{x \to a} g(x)\right)$$

iv) $$\lim_{x \to a} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$$

if $\lim_{x \to a} g(x) \neq 0$.

Example

$$\lim_{x\to 2} \frac{x^2+4x+5}{x^2+5}$$

$$\overset{(iv)}{=} \frac{\lim\limits_{x\to 2} x^2+4x+5}{\lim\limits_{x\to 2} x^2+5}$$

$$\underset{(ii)}{\overset{(i)}{=}} \frac{\lim\limits_{x\to 2} x^2 + 4\lim\limits_{x\to 2} x + \lim\limits_{x\to 2} 5}{\lim\limits_{x\to 2} x^2 + \lim\limits_{x\to 2} 5}$$

$$= \frac{4 + 8 + 5}{4+5} = \frac{17}{9}.$$

Sandwich Lemma

Suppose

$$f(x) \leq g(x) \leq h(x)$$

for all x near a (except possible $x = a$) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L.$$

Then

$$\lim_{x \to a} g(x) = L.$$

Example

Evaluate

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$$

Consider

$$g(x) = x^2 \sin\left(\frac{1}{x}\right).$$

$f(x) = -x^2$

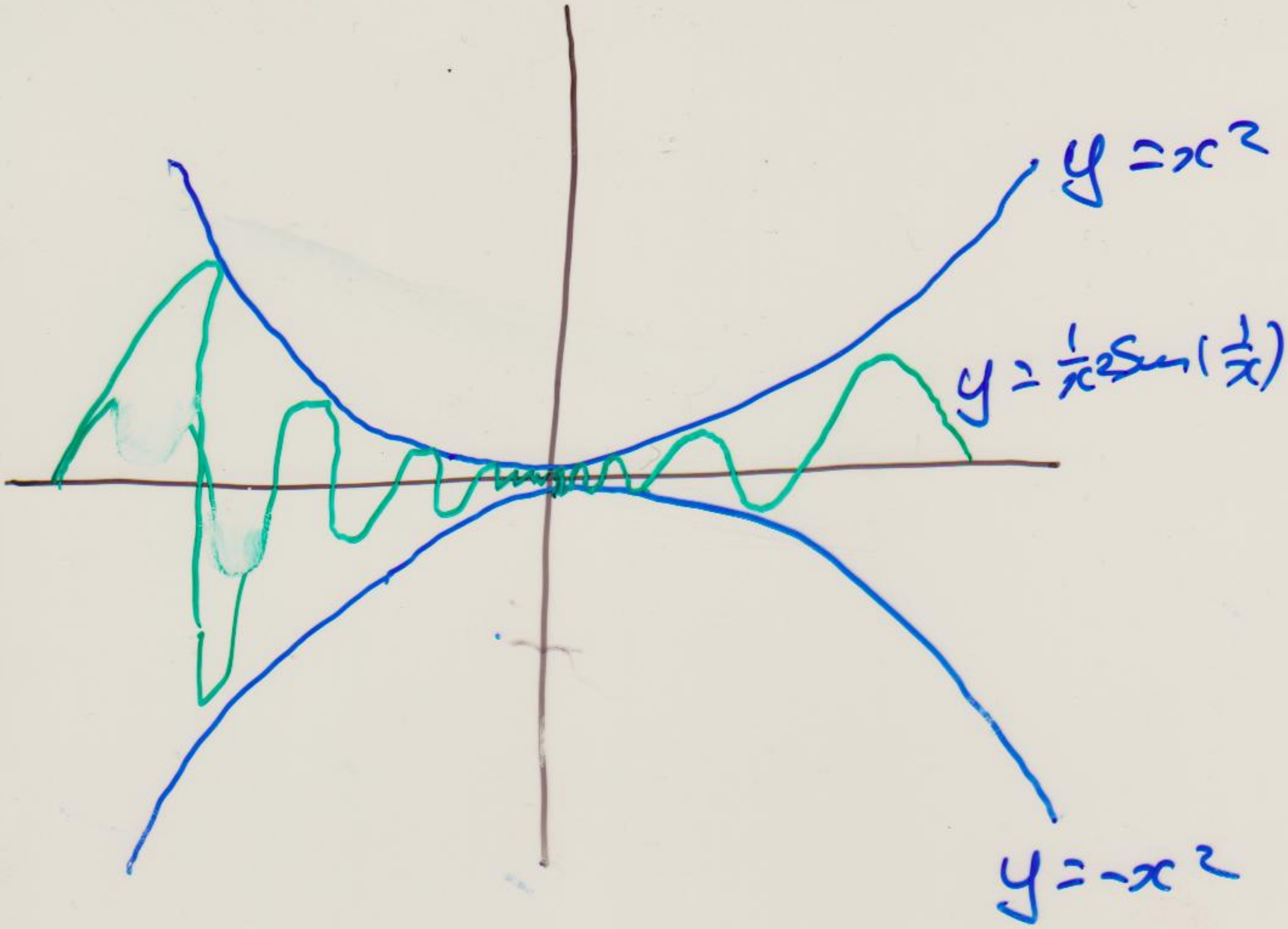
$h(x) = x^2$

$f(x) \leq g(x)$ for x near 0.

$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right)$

$g(x) \leq h(x)$

$x^2 \sin\left(\frac{1}{x}\right) \leq x^2$ for all x



$$-1 \le \sin\left(\frac{1}{x}\right) \le 1$$

$$-x^2 \le x^2 \sin\left(\frac{1}{x}\right) \le x^2$$

$$\lim_{x \to 0} -x^2 = 0$$

$$\lim_{x \to 0} x^2 = 0$$

By the Sandwich Lemma

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0.$$

Left-hand and right-hand limits.

We write
$$\lim_{x\to a-} f(x) = l$$
to mean that $f(x)$ is close to l for all x sufficiently close to a and to the left of a.

Example

Let $f(x) = \begin{cases} x^2+1 & \text{if } x \leq 3 \\ x+3 & \text{if } x > 3 \end{cases}$

$$\lim_{x \to 3-} f(x) = 10.$$

Analogously

$$\lim_{x \to 3+} f(x) = 11 .$$

Proposition

$$\lim_{x \to a} f(x) = L$$

if and only if

$$\lim_{x \to a-} f(x) = L \text{ and } \lim_{x \to a+} f(x) = L$$

Example

Let $f(x) = \begin{cases} x^2+1 & \text{if } x \le 3 \\ x + c & \text{if } x > 3 \end{cases}$

For what c does

$$\lim_{x \to 3} f(x)$$

exist?

Answer: $c = 7$ because

$$\lim_{x \to 3-} f(x) = 10$$

and we want

$$\lim_{x \to 3+} f(x) = x + c = 10.$$