Yesterday

$x(t)$ = world population at time t

rate of population growth proportional to population size

$$\boxed{\frac{dx}{dt} = k x}$$

exponential model

works well for small population size $x(t)$.

Gives silly population estimate for 2040.

Today:

$$\boxed{\frac{dx}{dt} = kx - lx^2}$$

logistic model

Here the constant l is tiny and lx^2 is only significant when x is large and x^2 is enormous.

suppose $F(x)$ is an antiderivative for

$$f(x) = \frac{1}{kx - lx^2}$$

Then

$$\boxed{\frac{dx}{dt} = kx - lx^2}$$ (1)

$$\Longleftrightarrow$$

$$\boxed{\frac{1}{kx - lx^2} \frac{dx}{dt} = 1}$$

$$\Longleftrightarrow$$

$$\boxed{F(x) = t}$$ (2)

We'd like to find an antiderivative $F(x)$ for $f(x) = \frac{1}{kx - lx^2}$

Suppose

$$f(x) = \frac{1}{x(k-lx)} = \frac{A}{x} + \frac{B}{k-lx}$$

with A, B constants.

Well, this would mean

$$\frac{1}{x(k-lx)} = \frac{A(k-lx)+Bx}{x(k-lx)}$$

Thus

$$1 = Ak + (B-Al)x$$

Need $B-Al=0$

$1 = Ak$

So $A = \frac{1}{k}$, $B = \frac{l}{k}$

In summary:

$$\boxed{f(x) = \frac{1}{kx} + \frac{l}{k^2-klx}}$$

Thus an anti derivative for $f(x)$ is

$$F(x) = \frac{1}{k}\ln(kx) + \frac{\cancel{l}}{-k\cancel{l}}\ln(k^2-klx)$$

$$F(x) = \frac{1}{k}\left\{ \ln(kx) - \ln(k^2 - klx) \right\}$$

$$F(x) = \frac{1}{k} \ln\left(\frac{kx}{k^2 - klx} \right)$$

$$F(x) = \frac{1}{k} \ln\left(\frac{x}{k - lx} \right)$$

Equation (2) becomes

$$t = \frac{1}{k} \ln\left(\frac{x}{k - lx} \right)$$

$$\Leftrightarrow \quad kt = \ln\left(\frac{x}{k - lx} \right)$$

$$\Leftrightarrow \quad e^{kt} = \frac{x}{k - lx}$$

$$\Leftrightarrow \quad (k - lx)\, e^{kt} = x$$

$\Leftrightarrow$

$$k e^{kt} = x(1 + l e^{kt})$$

$\Leftrightarrow$

$$x = \frac{k e^{kt}}{1 + l e^{kt}}$$

$$\longrightarrow \frac{k e^{kt}}{l e^{kt}} = \frac{k}{l}.$$

Conclusion: The logistic model implies that the world population $X(t)$ will tend to a constant value $\frac{k}{l}$ as time increases.

We can estimate k and l from our knowledge of the world population at various times.

An estimate (from Braun's 1970s book) is

$$\boxed{x(t) \to \frac{k}{l} \simeq 9.86 \text{ billion}}$$