Antiderivatives

A function $F(x)$ is called an **antiderivative** of a function $f(x)$ if

$$F'(x) = f(x)$$

<u>Example</u> An anti-derivative of $f(x) = \frac{1}{x+1}$ is

$$F(x) = \ln|x+1|.$$

<u>Example</u> An anti-derivative of $f(x) = \frac{1}{3} x e^{x^2}$ is

$$F(x) = \frac{1}{6} e^{x^2}$$

Lemma Any two antiderivatives of $f(x)$ differ by a constant.

Proof If $F(x)$ and $H(x)$ are both anti-derivatives of $f(x)$, then

$$\frac{d}{dx}\left(F(x) - H(x)\right)$$

$$= F'(x) - H'(x)$$

$$= f(x) - f(x)$$

$$= 0.$$

Hence $F(x) - H(x) = c$ is a constant. $\boxed{///}$

Notation If $F(x)$ is an antiderivative of $f(x)$ then we write

$$\int f(x)\,dx = F(x) + c.$$

Example

$$\int \cos(x)\,dx = \sin(x) + C$$

Example Observations suggest that the rate of growth of a population of zebra mussels is exponential in time.

Suppose

$$\frac{dN}{dt} = e^{2t}$$

where $N(t)$ is the number of mussels at time t (days).

If there are 150 mussels at time $t=0$, how many will there be after 10 days?

c) if $f(x) = \cos(x)$ then

$$F(x) = \sin(x) + C$$

d) if $f(x) = e^x$ then

$$F(x) = e^x + C$$

e) if $f(x) = \frac{1}{x}$ then

$$F(x) = \ln(x) + C$$

f) if F is an anti-derivative of f, and G is an anti-derivative g, then F+G is an anti-derivative of f+g.

<u>Example</u> The world's population was 2560 millions in 1950, and 3040 millions in 1960.

Assume that growth rate of the population is proportional to the size of the population.

(a) Estimate population in 1990

(b) " " " 2040

<u>Sol</u>

<u>Soln</u> Let $p(t)$ = population at time t (years).

$$\boxed{p'(t) = k\,p(t)} \quad *$$

where k is constant. So

$$\boxed{p(t) = A e^{kt}}$$

Sol[n] If $\frac{dN}{dt} = e^{2t}$ then

$$N(t) = \frac{1}{2}e^{2t} + c$$

$$N(0) = \frac{1}{2} + c = 150$$

Thus $c = 149.5$

Thus

$$\boxed{N(t) = \frac{1}{2}e^{2t} + 149.5}$$

At $t = 10$ days we have

$$N(10) = \frac{1}{2}e^{20} + 149.5 \simeq 2.4 \times 10^{8}$$

Mussels

Basic Rules of Antiderivatives

a) If $f(x) = x^n$ then $F(x) = \frac{x^{n+1}}{n+1} + c$

$(n \neq -1)$

a) If $f(x) = \sin(x)$ then

$$F(x) = -\cos(x) + c$$

$$p(0) = 2560 = Ae^0$$

$$A = 2560$$

So $\boxed{p(t) = 2560\, e^{kt}}$

$$p(10) = 3040 = 2560\, e^{k.10}$$

$$\frac{3040}{2560} = e^{k.10}$$

$$\frac{1}{10} \ln\left(\frac{3040}{2560}\right) = k \simeq 0.017185$$

At $t = 40$ (i.e. in 1990)

we have

$$p(40) = 2560\, e^{(0.017185) \times 40}$$

$$\simeq 5090 \text{ millions}$$

(Actual figure: 5278 millions)

At $t = 90$ (i.e. in 2040)

$p(90) = 12021$ millions.