Yesterday:

we showed that the function

$$y = e^{kt}$$

where k is a constant, satisfies the differential equation

$$\frac{dy}{dt} = ky \quad (*)$$

$$[y' = ky]$$

In fact

$$y = Ae^{kt}$$

satisfies the differential equation $(*)$, where A is any constant.

Are there any other functions that satisfy the differential equation (*) ?

Suppose $y = y(t)$ and $z = z(t)$ both satisfy (*).

Thus

$$\frac{d}{dt}\left(\frac{y}{z}\right) = \frac{z'y - y'z}{z^2}$$

$$= \frac{kzy - kyz}{z^2}$$

$$= 0.$$

So $\frac{y}{z}$ is a constant, say $\frac{y}{z} = A$. Hence $y = Az$

Conclusion:

The only solutions to the diff. eq^n. $(*)$ are

$$(+)\quad \boxed{y = A e^{kt} \qquad A \text{ constant}}$$

Example

A cup of coffee in a room at 20°C cools from 80°C to 50°C in five minutes. How long will it take to cool to 40°C ?

Sol^n

Newton: A hot object cools at a rate proportional to the excess of its temperature above room temperature.

$y(t)$ = temperature of coffee at time t

$y(0) = 80$

$y(5) = 50$

Question is: for what t do we have $y(t) = 40$.

Newton:

$$\frac{dy}{dt} = k(y-20)$$

Consider $z = y-20$

$z(0) = 60$

$z(5) = 30$

$$\frac{dz}{dt} = \frac{d}{dt}(y-20) = \frac{dy}{dt}$$

$$= k(y-20) = kz$$

Hence $\boxed{\frac{dz}{dt} = kz}$ *

Our question is:
For what of t do we have $z(t) = 40-20 = 20$?

Since $z(t)$ satisfies the diff. eqn (*) we know that

$$z = Ae^{kt}$$

for some constants A, k.

$$60 = z(0) = Ae^{k.0} = Ae^0 = A$$

$$\boxed{A = 60}$$

so $z = 60e^{kt}$

$$30 = z(5) = 60e^{k.5}$$

$$\boxed{\frac{1}{2} = e^{5k}}$$

For what t do we have

$z(t) = 20$?

$$20 = z(t) = 60 e^{kt}$$

$$\frac{1}{3} = e^{kt}$$

$$\frac{1}{3} = \left(e^{5k}\right)^{\frac{t}{5}}$$

$$\frac{1}{3} = \left(\frac{1}{2}\right)^{\frac{t}{5}}$$

$$\ln\left(\frac{1}{3}\right) = \ln\left(\left(\frac{1}{2}\right)^{t/5}\right)$$

$$\ln\left(\frac{1}{3}\right) = \frac{t}{5} \ln\left(\frac{1}{2}\right)$$

$$t = \frac{5 \ln\left(\frac{1}{3}\right)}{\ln\left(\frac{1}{2}\right)} \text{ minutes}$$