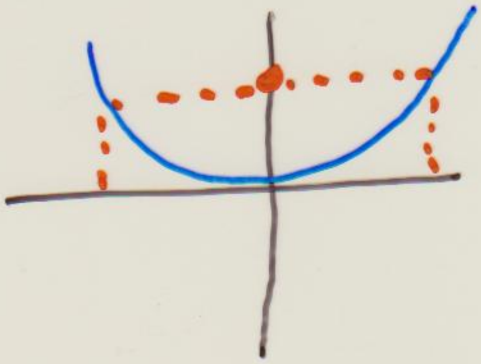
A function $f: D \to \mathbb{R}$ is said to be <u>INJECTIVE</u> if

$$f(x_1) \neq f(x_2) \text{ for } x_1 \neq x_2 \in D.$$

<u>Example</u> a) $f(x) = x^2$

This is <u>not</u> injective because $f(1) = f(-1)$ and $1 \neq -1$.



b) $f(x) = x^3 - 2$

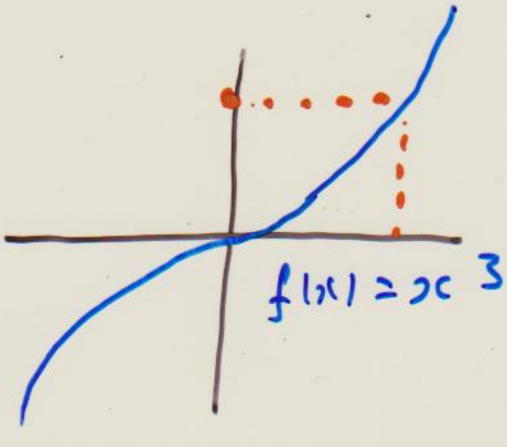
$f(x_1) = f(x_2)$

$\Rightarrow x_1^3 - 2 = x_2^3 - 2$

$\Rightarrow x_1^3 = x_2^3$

$\Rightarrow x_1 = x_2$

So $f(x)$ <u>is</u> injective.



Suppose that f is an injective function. Then the INVERSE FUNCTION f^{-1} is defined by the rule

$$f^{-1}(y) = x \text{ exactly when } y = f(x).$$

Example Find the inverse of $f(x) = x^5 - 14$.

$$y = x^5 - 14$$

$$y + 14 = x^5$$

$$\sqrt[5]{y+14} = x$$

So $f^{-1}(x) = \sqrt[5]{x+14}$

Observe that

$$\boxed{f^{-1}(f(x)) = x} \qquad (a)$$

Proposition (Derivative of inverse function)

Suppose
$$f'(f^{-1}(x)) \neq 0.$$

Then
$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

Proof Differentiate (a) above.

$$(f^{-1})'(f(x)) \cdot f'(x) = 1$$

$$(f^{-1})'(f(x)) = \frac{1}{f'(x)}$$

now write $y = f(x)$ or $x = f^{-1}(y)$.

we get

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}.$$

$\square$

<u>Example</u> if $f(x) = x^5 - 14$

then

$$f^{-1}(x) = \sqrt[5]{x+14}$$

so from the proposition

$$(f^{-1})'(x) = \frac{1}{5(\sqrt[5]{x+14})^4}.$$

check!

N.B. $y = \ln(x)$ is injective.

So it has an inverse function which we write as

$$x \longmapsto e^x$$

DIFFERENTIAL EQUATIONS

An equation involving a derivative

$$\frac{dy}{dt} = k\,y \qquad (*)$$

where k is some constant and y is some function of t.

1) Are there any solutions?

Consider for instance

$$y = e^{kt}.$$

Is this a solution?

$$\frac{dy}{dt} = \frac{d}{dt}\left(e^{kt}\right) = e^{kt}.k = ky$$

Also,

$$y = 7e^{kt} \text{ is a solution.}$$

$$\frac{dy}{dt} = \frac{d}{dt}\left(7e^{kt}\right) = 7e^{kt} \cdot k = ky$$

We see that

$$y = Ae^{kt}$$

is a solution to

$$\frac{dy}{dt} = kt$$

for any constant A.