Logarithms done proper

Yesterday:

$y = \log_a x \iff a^y = x$

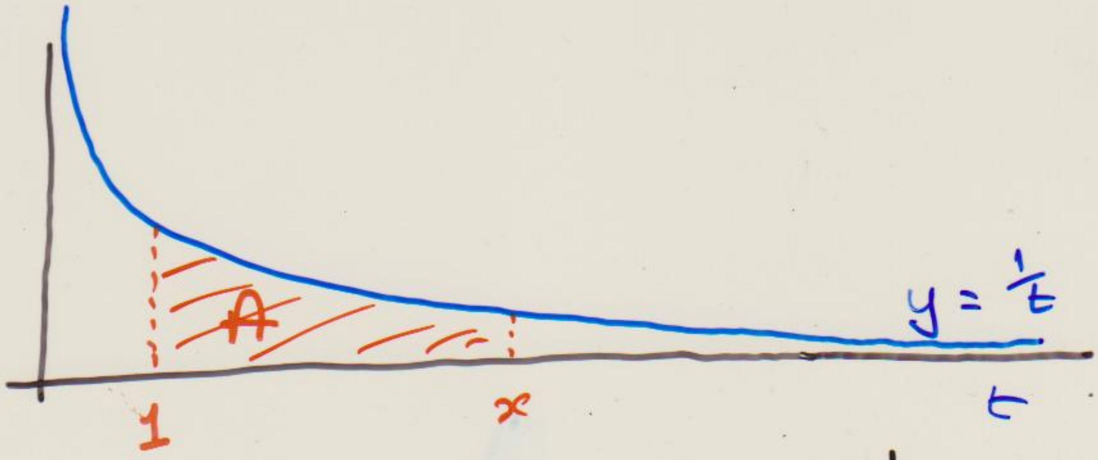
$\sqrt{2} = \log_4 ?$ $\qquad 4^{\sqrt{2}} = ?$

$$\boxed{\log_a(xy) = \log_a x + \log_a y}$$

Today's aim: explain rigorously what a logarithm is.

Definition For $x > 0$ let A

be the area



between the curve $y = \frac{1}{t}$ and the t-axis from $t = 1$ to $t = x$.

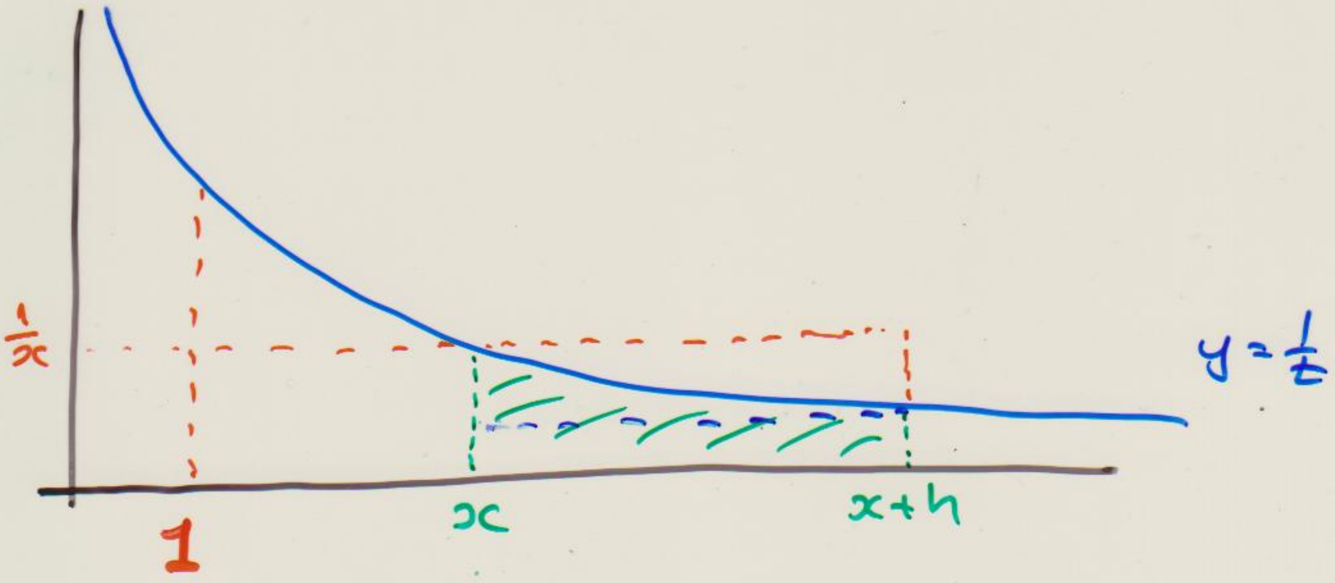
we define

$$\ln(x) = \begin{cases} A & \text{if } x \geq 1 \\ -A & \text{if } 0 < x < 1 \end{cases}$$

Theorem if $x > 0$ then

$$\frac{d}{dx} \ln(x) = \frac{1}{x}$$

Proof



$$\frac{h}{x+h} < \text{shaded area} < \frac{h}{x}$$

So, dividing by h

$$\frac{1}{x+h} < \frac{\ln(x+h) - \ln(x)}{h} < \frac{1}{x} \qquad (*)$$

So

$$\frac{d}{dx}\ln(x) = \lim_{h\to 0} \frac{\ln(x+h) - \ln(x)}{h}$$

from $(*)$ $= \frac{1}{x}$.

$$\boxed{\frac{d}{dx}\ln(x) = \frac{1}{x}}$$

<u>Consequence 1</u> For $x, y > 0$ we have

$$\ln(xy) = \ln(x) + \ln(y).$$

<u>Proof</u> $\frac{d}{dx}\left(\ln(xy) - \ln(x)\right)$

$$= \frac{\cancel{y}}{x\cancel{y}} - \frac{1}{x} = 0$$

Thus

$$\ln(xy) - \ln(x) = c \qquad (*)$$

c is a constant
not depending
on x

Put for example $x = 1$ into $(*)$.

Then

$$\ln(y) - \ln(1) = c.$$

From our definition of $\ln(x)$
we see that $\ln(1) = 0$.

So $c = \ln(y)$.

From $(*)$ we get

$$\ln(xy) - \ln(x) = \ln(y).$$

or

$$\ln(xy) = \ln(x) + \ln(y).$$

QED

We could also show that

$$\boxed{\ln(x^k) = k\ln(x).}$$

Past exam question:

Find the derivative y' of

$$y = \frac{(x+1)(x+2)(x+3)}{(x+4)}$$

Sol^n

$$\ln(y) = \ln\left((x+1)(x+2)(x+3)(x+4)^{-1}\right)$$

$$= \ln(x+1) + \ln(x+2) + \ln(x+3) - \ln(x+4)$$

Differentiate both sides with respect to x.

$$\frac{1}{y}y' = \frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} - \frac{1}{x+4}$$

$$y' = \frac{(x+1)(x+2)(x+3)}{(x+4)}\left\{\frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} - \frac{1}{x+4}\right\}$$