Some important terminology

- $f(x)$ is **continuous** at $x = c$ if $f(c)$ is defined and $\lim_{x \to c} f(x) = f(c)$.

- $f(x)$ is **differentiable** at $x = c$ if $f(c)$ is defined and the limit $f'(c)$ exists.

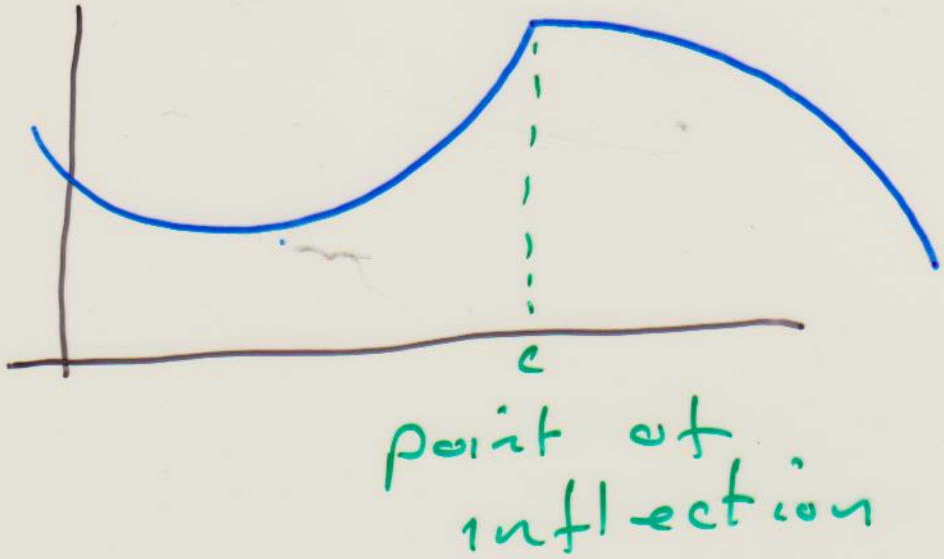
- $x = c$ is a **critical point** of $f(x)$ if either $f'(c)$ does not exist or $f'(c) = 0$.

- $f(x)$ is **increasing** on (a, b) if $f'(x) \geq 0$ for $x \in (a, b)$.

- $f(x)$ is **concave up** on (a, b) if $f''(x) \geq 0$ for $x \in (a, b)$.

(Informally: concave up $\Leftrightarrow$ accelerating)

• $x = c$ is a point of inflection of $f(x)$ if the concavity changes.



Some Important Theorems

Theorem If $f(x)$ is differentiable at $x = c$ then $f(x)$ is continuous at $x = c$.

Proof $f(x)$ differentiable at $x = c$

$$\Longleftrightarrow \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = l \text{ exists}$$

$$\Rightarrow \left(\lim_{h\to 0} h\right)\left(\lim_{h\to 0} \frac{f(c+h)-f(c)}{h}\right) = \left(\lim_{h\to 0} h\right).l$$

$$\Rightarrow \lim_{h\to 0} \cancel{h}\, \frac{f(c+h)-f(c)}{\cancel{h}} = 0$$

$$\Rightarrow \lim_{h\to 0} f(c+h) - \lim_{h\to 0} f(c) = 0$$

$$\Rightarrow \lim_{h\to 0} f(c+h) = f(c)$$

$$\Rightarrow \lim_{x\to c} f(x) = f(c)$$

$\Leftrightarrow$ $f(x)$ is continuous at $x=c$.

$\boxed{///}$

<u>Example</u> Find all possible values of a, b such that

$$f(x) = \begin{cases} x^2+x+1 & , x \geq 1 \\ ax+b & , x < 1 \end{cases}$$

is differentiable at all points.

Soln $f(x)$ is clearly differentiable at all points $x \neq 1$.

We need to choose a, b such that $f(x)$ is differentiable at $x = 1$.

In particular, $f(x)$ must be continuous at $x = 1$. i.e. need

$$\lim_{x \to 1} f(x) = f(1)$$

$$\Longleftrightarrow$$

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1+} f(x) = f(1)$$

$$\Longleftrightarrow$$

$$\boxed{a + b = 3} = 3$$

so choose $a + b = 3$.

now

$f'(1)$ exists

$\iff$

$\lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$ exists

$\iff$

$$\lim_{h \to 0-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0+} \frac{f(1+h) - f(1)}{h}$$

$$\boxed{\begin{aligned} p(x) &= x^2 + x + 1 \\ q(x) &= ax + b \end{aligned}}$$

$\iff$

$$\lim_{h \to 0-} \frac{q(1+h) - q(1)}{h} = \lim_{h \to 0+} \frac{p(1+h) - p(1)}{h}$$

$\iff$

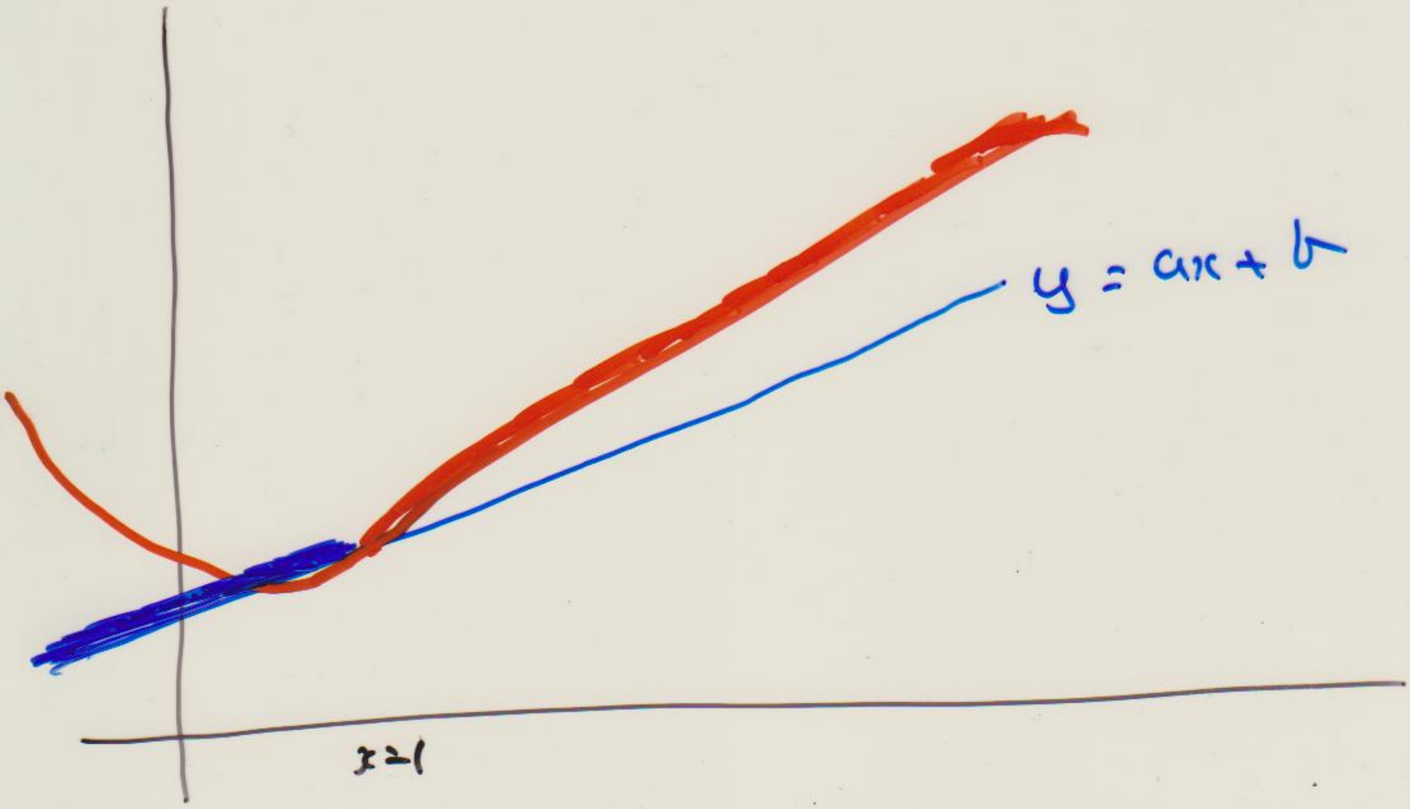
$$\boxed{q'(1) = p'(1)}$$ important line

$\iff$

$$\boxed{a = 3}$$

In summary, $f(x)$ is differentiable at $x=1$ if and only if:

$$\boxed{\begin{gathered} a+b=3 \\ \text{and} \\ a=3 \end{gathered}} \quad \text{so } b=0.$$

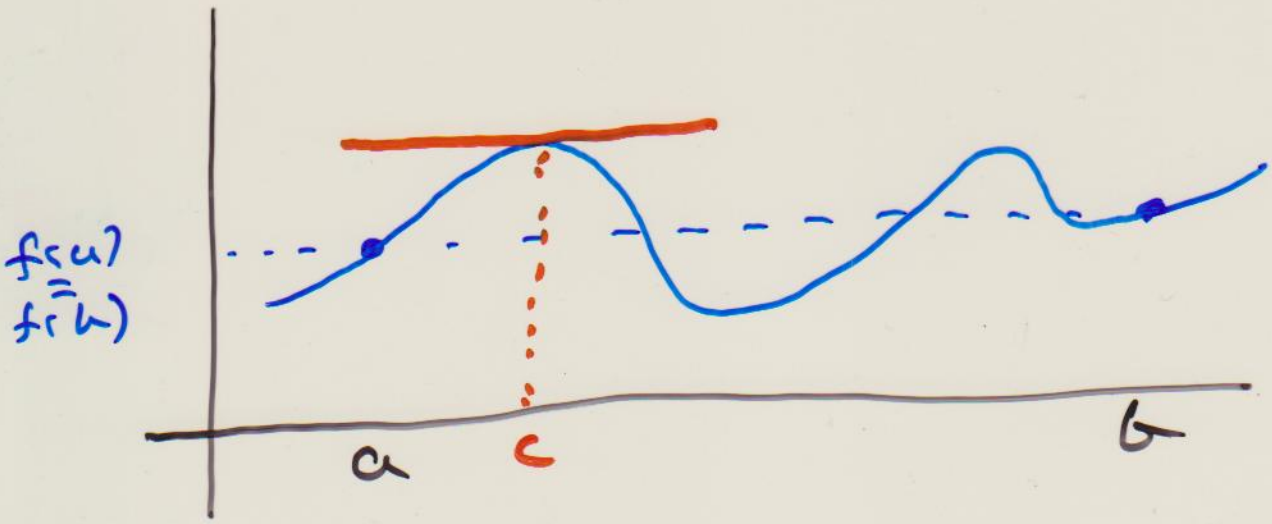


Rolle's Theorem

suppose $f(x)$ is :

- is continuous on $[a, b]$
- and differentiable on (a, b)
- and $f(a) = f(b)$.

Then there exist at least one point $c \in (a, b)$ such that $f'(c) = 0$.



Example Prove that there is exactly one solution to $x^3 + x + 1 = 0$.

Let $f(x) = x^3 + x + 1$

$f(-1) < 0$

$f(0) > 0$

so by IVT there is at least one $c \in [-1, 0]$ such that $f(c) = 0$.

$f'(x) = 3x^2 + 1 > 0$ for all x.

So Rolle's Theorem implies that there is no other solution to $f(x) = 0$.