Yesterday

Defn 1 The Golden Ratio is

$$\phi = \frac{1+\sqrt{5}}{2}$$

Defn 2 Two numbers $a > b > 0$ are in the Golden Ratio if

$$\frac{a+b}{a} = \frac{a}{b}$$

N.B. if $\frac{a+b}{a} = \lambda$ and $\frac{a}{b} = \lambda$

then $a = b\lambda$ and

$$\frac{b\lambda + b}{b\lambda} = \lambda$$

or $\frac{\lambda+1}{\lambda} = \lambda$

or $\lambda^2 - \lambda - 1 = 0$.

so $\lambda = \frac{1+\sqrt{1+4}}{2}$

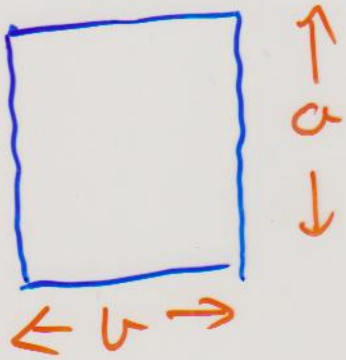
or $\lambda = \frac{1-\sqrt{1+4}}{2}$

Since $\frac{a}{b} > 0$ we have $\lambda = \frac{1+\sqrt{5}}{2}$.

Belief: A beautiful body is such that

with $\frac{a}{b} = \Phi$

Belief A beautiful window has the form



with $\frac{a}{b} = \Phi$

Defn 3 The Golden Ratio Φ is the positive eigenvalue of

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

$$\det(A - \lambda I) = 0$$

$$\det \begin{pmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = 0$$

$$(1-\lambda)(-\lambda) - 1 = 0$$

$$\lambda^2 - \lambda - 1 = 0$$

so $\lambda = \dfrac{1+\sqrt{5}}{2} > 0$

or $\lambda = \dfrac{1-\sqrt{5}}{2} < 0$

Today:

Given

$F_0 = 0,\ F_1 = 1,\ F_2 = 1,\ \ldots$

where $F_n = F_{n-1} + F_{n-2}$

want to find a closed formula for F_n.

<u>Theorem</u> If a 2×2 matrix A has eigenvalues λ_1, λ_2 with corresponding eigenvectors v_1, v_2, and if the matrix $T = \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix}$ is invertible, then

$$T^{-1} A T = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Proof

$$T^{-1}AT = T^{-1}A\begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix}$$

$$= T^{-1}\begin{pmatrix} | & | \\ \lambda_1 v_1 & \lambda_2 v_2 \\ | & | \end{pmatrix}$$

$$= T^{-1}\begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$= I\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

QED

Example

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

has eigenvalues

$$\phi = \frac{1+\sqrt{5}}{2}$$

$$\bar{\phi} = \frac{1-\sqrt{5}}{2}.$$

Let's find eigenvectors v_1, v_2.

Need to solve

$$(A - \lambda I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{for } \begin{array}{l} \lambda = \phi \\ \lambda = \bar{\phi} \end{array}$$

$\lambda = \phi$

$$\begin{pmatrix} \frac{1-\sqrt{5}}{2} & 1 \\ 1 & \frac{-1-\sqrt{5}}{2} \end{pmatrix} \underbrace{\begin{pmatrix} \phi \\ 1 \end{pmatrix}}_{\substack{\text{eigenvector} \\ \text{for } \lambda = \phi}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\lambda = \bar{\phi}$

$$\begin{pmatrix} \frac{1+\sqrt{5}}{2} & 1 \\ 1 & \frac{-1+\sqrt{5}}{2} \end{pmatrix} \underbrace{\begin{pmatrix} 1 \\ -\phi \end{pmatrix}}_{\substack{\text{eigenvector} \\ \text{for } \lambda = \bar{\phi}}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Set

$$T = \begin{pmatrix} \phi & 1 \\ 1 & -\phi \end{pmatrix}.$$

Then

$$\boxed{T^{-1} A T = \begin{pmatrix} \phi & 0 \\ 0 & \bar{\phi} \end{pmatrix}}$$

Recall: Our sequence

$F_0 = 0,\ F_1 = 1,\quad F_n = F_{n-1} + F_{n-2}$

can be written

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \overbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}^{A} \begin{pmatrix} F_{n-1} \\ F_{n-2} \end{pmatrix}$$

$$= A\, A \begin{pmatrix} F_{n-2} \\ F_{n-3} \end{pmatrix}$$

$$= A^2 \begin{pmatrix} F_{n-2} \\ F_{n-3} \end{pmatrix}$$

$$\vdots$$

$$= A^{n-1} \begin{pmatrix} F_1 \\ F_0 \end{pmatrix}$$

$$= A^{n-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \, .$$

Now

$$T^{-1} A T = \begin{pmatrix} \phi & 0 \\ 0 & \bar{\phi} \end{pmatrix}$$

and

$$A = T \overbrace{\begin{pmatrix} \phi & 0 \\ 0 & \bar{\phi} \end{pmatrix}}^{D} T^{-1}$$

$$A^n = (TD\cancel{T^{-1}})(\cancel{T}D\cancel{T^{-1}})(\cancel{T}D\cancel{T^{-1}}) \cdots (\cancel{T}DT^{-1})$$

$$= T D^n T^{-1}$$

$$= T \begin{pmatrix} \phi & 0 \\ 0 & \bar{\phi} \end{pmatrix}^n T^{-1}$$

$$= T \begin{pmatrix} \phi^n & 0 \\ 0 & \bar{\phi}^n \end{pmatrix} T^{-1}$$

Conclusion :

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = A^{n-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= T \begin{pmatrix} \phi^{n-1} & 0 \\ 0 & \bar{\phi}^{n-1} \end{pmatrix} T^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{1}{-\phi^2-1} \begin{pmatrix} \phi & 1 \\ 1 & -\phi \end{pmatrix} \begin{pmatrix} \phi^{n-1} & 0 \\ 0 & \bar{\phi}^{n-1} \end{pmatrix} \begin{pmatrix} -\phi & -1 \\ -1 & \phi \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Exercise: from this we see

$$\boxed{F_n = \frac{1}{\sqrt{5}} \phi^n - \frac{1}{\sqrt{5}} \bar{\phi}^n}$$