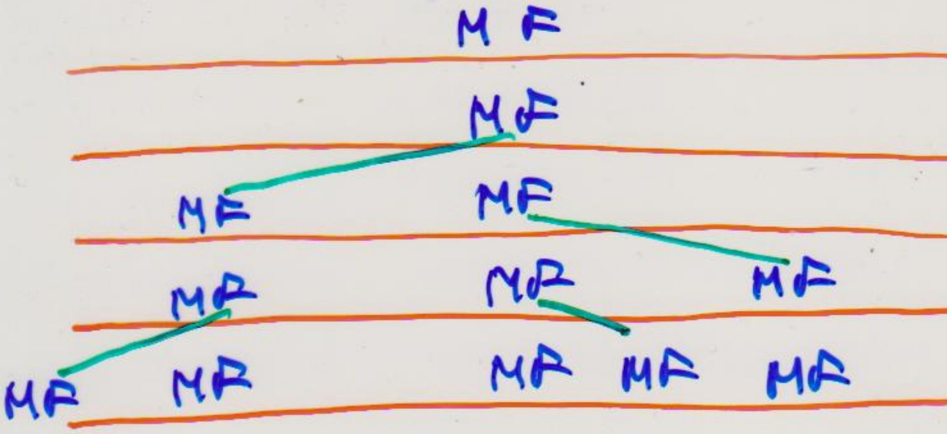
Rabbits

- One newly born male rabbit and one newly born female rabbit are placed in a field.
- Rabbits can mate at 1 month and one month later the female produces one male/female pair.
- Rabbits don't die

How fast does the rabbit population grow?



t	number of pairs
0	1
1	1
2	2
3	3
4	5
5	8

At $t=12$ how many rabbits are there?

Let F_n = number of pairs of rabbits after n months

("F" is for Fibonacci)

$F_0 = 1$

$F_1 = 1$

$F_2 = F_1 + F_0$

$F_3 = F_2 + F_1$

$F_4 = F_3 + F_2$

Generally

$$\boxed{F_n = F_{n-1} + F_{n-2}}$$

1, 1, 2, 3, 5, 8, 13, 21, 34, 55

89, 144, 233

F_{12}

$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21},$

$\underbrace{\frac{55}{34}}_{1.617647}, \frac{89}{55}, \frac{144}{89}, \underbrace{\frac{233}{141}}_{1.6180555}, \ldots\ldots$

Maybe this sequence F_{n+1}/F_n converges.

Maybe

$$\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \phi$$

where ϕ is some number near 1.618...)

If this limit ϕ exists, then population increases by (roughly) a factor of ϕ each month.

$$F_n = F_{n-1} + F_{n-2}$$

$$F_{n-1} = F_{n-1}$$

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{n-1} \\ F_{n-2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{n-2} \\ F_{n-3} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^2 \begin{pmatrix} F_{n-2} \\ F_{n-3} \end{pmatrix}$$

$$\vdots$$

$$\boxed{\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} F_1 \\ F_0 \end{pmatrix}}$$

If the limit ϕ exists then for large n

$$\frac{F_n}{F_{n-1}} \simeq \phi, \text{ or } F_n \simeq \phi F_{n-1}$$

or

$$\underbrace{\phi \begin{pmatrix} F_{n-1} \\ F_{n-2} \end{pmatrix}} \simeq \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \underbrace{\overbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}^{A} \begin{pmatrix} F_{n-1} \\ F_{n-2} \end{pmatrix}}$$

So ϕ would be (roughly) an eigenvalue for

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

To find the eigenvalues of A:

$$\det \begin{pmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = 0$$

$$(1-\lambda)(-\lambda) - 1 = 0$$

$$\lambda^2 - \lambda - 1 = 0$$

Eigenvalues for A are

$$\lambda_1 = \frac{1+\sqrt{1+4}}{2} = \frac{1+\sqrt{5}}{2} \simeq 1.618\ldots$$

$$\lambda_2 = \frac{1-\sqrt{1+4}}{2} = \frac{1-\sqrt{5}}{2}$$

The number

$$\phi = \frac{1+\sqrt{5}}{2}$$

is called the Golden Ratio

Alternative Definition Two quantities $b > a$ are said to be in the Golden Ratio if

$$\frac{a+b}{b} = \frac{b}{a}.$$