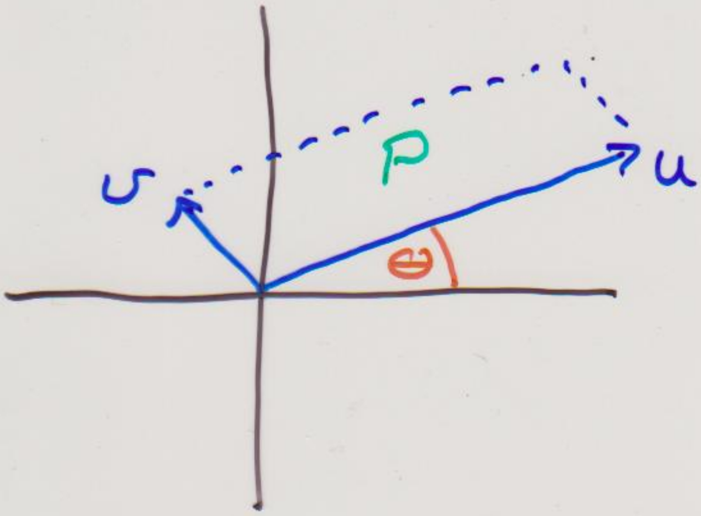
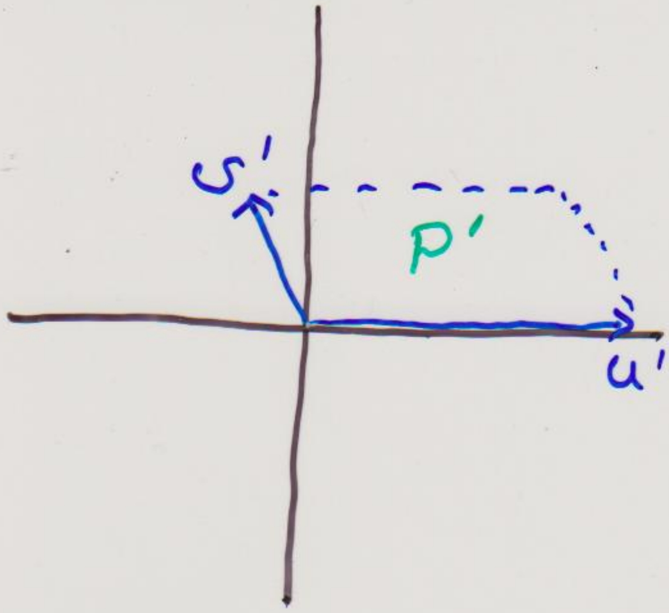




Recall

$$u = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} u'$$

$$v = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} v'$$

$$\det \begin{pmatrix} | & | \\ u & v \\ | & | \end{pmatrix}$$

$$= \det \left(\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} | & | \\ u' & v' \\ | & | \end{pmatrix} \right)$$

$$= \det \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \det \begin{pmatrix} | & | \\ u' & v' \\ | & | \end{pmatrix}$$

$= (\cos^2\theta + \sin^2\theta) . (\pm \text{ area } P')$

$= \pm \text{ area } P.$

QED

Eigenvalues & Eigenvectors

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 matrix of real numbers.

Defn A non-zero vector $v = \begin{pmatrix} x \\ y \end{pmatrix}$ is called an eigenvector for A if there exists a real number λ such that

$$A v = \lambda v.$$

We call λ the corresponding eigenvalue.

Example

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Consider

$$v = \begin{pmatrix} 4 \\ 4 \end{pmatrix}.$$

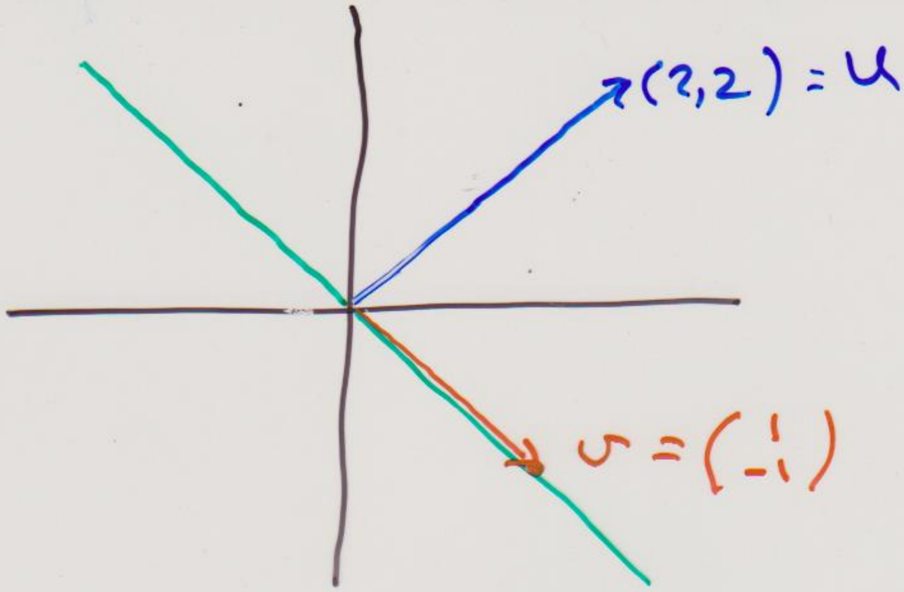
Then

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 4 \\ 4 \end{pmatrix} = \begin{pmatrix} 12 \\ 12 \end{pmatrix}.$$

so $$3. \begin{pmatrix} 4 \\ 4 \end{pmatrix} = \begin{pmatrix} 12 \\ 12 \end{pmatrix}$$

Thus $v = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$ is an eigenvector of A with eigenvalue $\lambda = 3$.

<u>Example</u> Let A be the matrix of reflection in the line $y = -x$.



$Av = v$

so $v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector

with eigenvalue $\lambda = 1$.

$Au = -u$

so $u = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ is an eigenvector

with eigenvalue $\lambda = -1$.

<u>Example</u> Let A be the matrix of anticlockwise rotation about the origin through $45°$.

This matrix has no eigenvectors.

Proposition Let A be a 2×2 matrix, let $v = \begin{pmatrix} x \\ y \end{pmatrix}$ be a non-zero vector, and suppose that

$$A v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Then $\det(A) = 0$.

Proof

If A^{-1} existed then

$$A^{-1} A v = A^{-1} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and

$$v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus A^{-1} does not exist.

$\left(A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) \right)$

So $\det(A) = 0$. QED

How can we find all eigenvalues of

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} ?$$

Suppose v is some eigenvector with eigenvalue λ.

$$Av = \lambda v .$$

Then

$$Av - \lambda v = \begin{pmatrix} 0 \\ 0 \end{pmatrix} .$$

Then

$$(A - \lambda I) v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So $\det(A - \lambda I) = 0$.

$$A - \lambda I = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix}$$

$\det(A - \lambda I) =$

$$(2-\lambda)(2-\lambda) - 1.1$$

$$= \lambda^2 - 4\lambda + 3$$

$$= (\lambda - 3)(\lambda - 1)$$

$\lambda = 1, 3$

To find eigenvectors we need

$$Av = \lambda v$$

$$(A - \lambda I)v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Must solve

$$\begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

to find eigenvectors.

$\lambda = 1$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So, for instance

$v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector for $\lambda = 1$.

$\lambda = 3$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So, for instance,

$v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector for $\lambda = 3$.