Mon 12th Dec 2011

10 am - 12 am

MA 180/190 Christmus Test

Topic 3

Determinants, Eigenvalues & Eigenvectors

(Mainly 2×2 case)

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Definition The adjoint matrix is

$$\mathrm{adj}(A) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Observe

$$A.\,\mathrm{adj}(A) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$= \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} = (ad-bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Definition The determinant of A is the number

$$\det(A) = ad - bc.$$

Also write

$$|A| = ad - bc.$$

Example

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$\det(A) = \underline{\underline{1.4 - 3.2 = -2}}$$

Note:

$$A . \operatorname{adj}(A) = \det(A) . I$$

or

$$A \left(\frac{1}{\det(A)} \cdot \operatorname{adj}(A) \right) = I.$$

"Thus"

Proposition if $|A| \neq 0$ then

$$A^{-1} = \frac{1}{|A|} \operatorname{adj}(A) .$$

Example Solve

$$3x + y = 2$$
$$4x + 2y = 4$$

Sol^n

$$\underbrace{\begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

So
$$\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1} \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} .$$

So $x = 0$, $y = 2$.

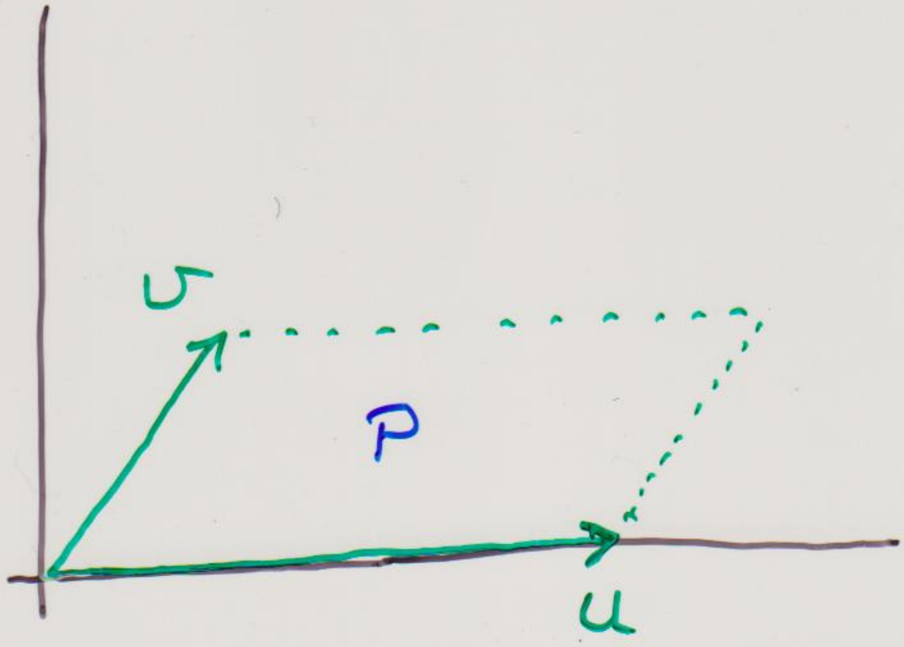
Proposition (Exercise)

If $|A| = 0$ then A has no inverse.

Determinants are related to areas of parallelograms.

Consider "random" vectors

$$u = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \qquad v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$



So u, v determine a parall$^{\text{m}}$ P.

<u>Proposition</u> Given two 2+2 matrices A, B we have

$$|AB| = |A||B|.$$

<u>Example</u>

Consider

$$A = \begin{pmatrix} 3 & 1 \\ 7 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 8 \\ 9 & 6 \end{pmatrix}$$

$$AB = \begin{pmatrix} 15 & 30 \\ 50 & 80 \end{pmatrix}$$

$$\begin{aligned} |AB| &= 15.80 - 50.30 \\ &= 10(15.8 - 50.3) \\ &= 50(3.8 - 10.3) \\ &= 50(-6) = -300 \end{aligned}$$

$|A| = 5 \qquad |B| = -60$

Note: $|A||B| = -300 = |AB|$

Area of P = base $\times$ $\perp^r$ height

$$= 3 \times 2$$

$$= 6.$$

$A = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$ so the vectors u, v also determine a matrix A.

$$|A| = 3.2 - 0.1 = 6$$

<u>Theorem</u> The determinant of a 2×2 matrix is equal to $\pm$ area of the parallelogram determined by its two columns.