Inverting a matrix

How can we find the inverse of a matrix such as

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 8 & 6 \end{pmatrix} ?$$

That is, how can we find a matrix B such that

$$AB = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} ?$$

Gauss - Jordan method for finding the inverse of a matrix A

$$(A \mid I) \xrightarrow[\text{operations}]{\text{row}} (I \mid B)$$

Then $B = A^{-1}$.

There are three kinds of row operation:

(I) $R_i \longmapsto R_i + \lambda R_j \qquad (j \neq i)$

(II) $R_i \longleftrightarrow R_j \qquad (j \neq i)$

(III) $R_i \longrightarrow \lambda R_i \qquad (\lambda \neq 0)$

Illustration of above matrix A.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 5 & 0 & 1 & 0 \\ 3 & 8 & 6 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3 \mapsto R_3 - 3R_1]{R_2 \mapsto R_2 - 2R_1}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 2 & -3 & -3 & 0 & 1 \end{array}\right) \xrightarrow{R_3 \mapsto R_3 - 2R_2}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -1 & 1 & -2 & 1 \end{array}\right) \xrightarrow{R_3 \mapsto -R_3}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right) \xrightarrow[R_2 \mapsto R_2 + R_3]{R_1 \mapsto R_1 - 3R_3}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 4 & -6 & 3 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right) \xrightarrow{R_1 \mapsto R_1 - 2R_2}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 10 & -12 & 5 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right)$$

So

$$A^{-1} = \begin{pmatrix} 10 & -12 & 5 \\ -3 & 3 & -1 \\ -1 & 2 & -1 \end{pmatrix}$$

Check

$$\underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 8 & 6 \end{pmatrix}}_{A} \begin{pmatrix} 10 & -12 & 5 \\ -3 & 3 & -1 \\ -1 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

So answer for A^{-1} is correct.

Towards: why does the method work.

operation I

e.g. $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 8 & 6 \end{pmatrix} \xrightarrow{R_2 \mapsto R_2 - 2R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 3 & 8 & 6 \end{pmatrix}$

Also

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 8 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 3 & 8 & 6 \end{pmatrix}$$

In general

$$A \xrightarrow{R_i \mapsto R_i + \lambda R_j} B$$

then

$$E_{ij}^{\lambda} A = B$$

where E_{ij}^{λ} has : 1 on the diagonals

: λ in i^{th} row and j^{th} column

: 0 elsewhere

Operation II

e.g. $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 8 & 6 \end{pmatrix} \xrightarrow{R_2 \Leftrightarrow R_3} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 8 & 6 \\ 2 & 5 & 5 \end{pmatrix}$

Also

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 8 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 8 & 6 \\ 2 & 5 & 5 \end{pmatrix}$$

In general,

$$A \xrightarrow[(i \neq j)]{R_i \Leftrightarrow R_j} B$$

then

$$P_{ij} A = B$$

where P_{ij} has:

1 in i^{th} row and j^{th} column
1 in j^{th} row and i^{th} column
1 in diagonal entries of all other rows

0 elsewhere.