Yesterday: we checked that

$$T: \mathbb{R}^2 \to \mathbb{R}^2, \ (x,y) \mapsto (x+2y,\ 3x+4y)$$

was linear.

Note that

$$T(x,y) = (x+2y,\ 3x+4y)$$

can be represented as matrix multiplication

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+2y \\ 3x+4y \end{pmatrix}$$

we say that

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

represents T.

<u>Theorem</u> Any linear transformation T can be represented by a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

<u>Proof</u>

Well $T(1,0) = (a,b)$ say.

and

$T(0,1) = (c,d)$ say.

Then

$T(x,y) =$

$T\big(x(1,0) + y(0,1)\big)$

$= T\big(x(1,0)\big) + T\big(y(0,1)\big)$

$= x\,T(1,0) + y\,T(0,1)$ By linearity of T

$= x(a,b) + y(c,d)$

$= (ax+cy, bx+dy)$

Now

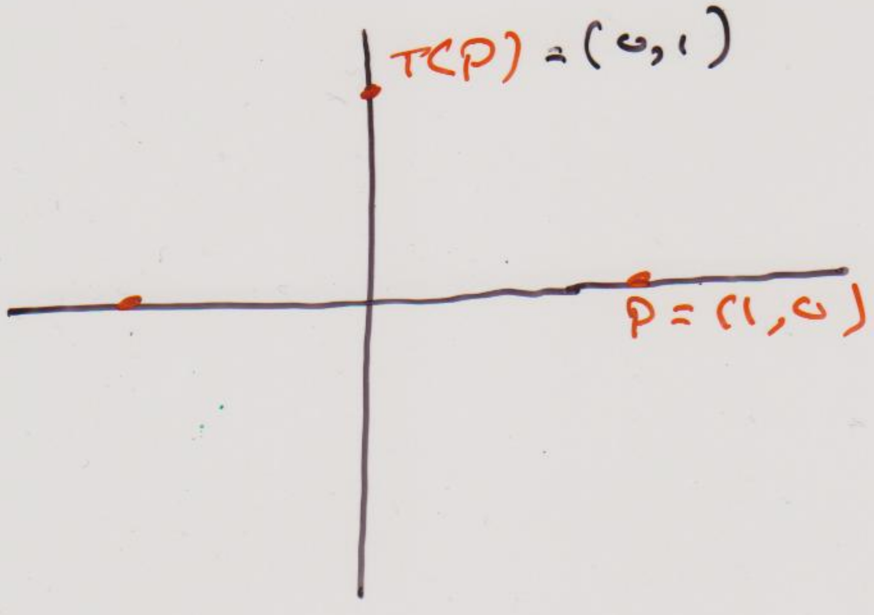
$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+cy \\ bx+dy \end{pmatrix}.$$

FACTS: Any reflection is linear.

Any rotation is linear.

Any composite of a reflection and rotation is linear.

Example Find the matrix representing a reflection in the y-axis, followed by a clockwise rotation of $\frac{5\pi}{2}$ rads about the origin.



So $T(1,0) = (0,1)$

$T(0,1) = (1,0)$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

So the required matrix is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

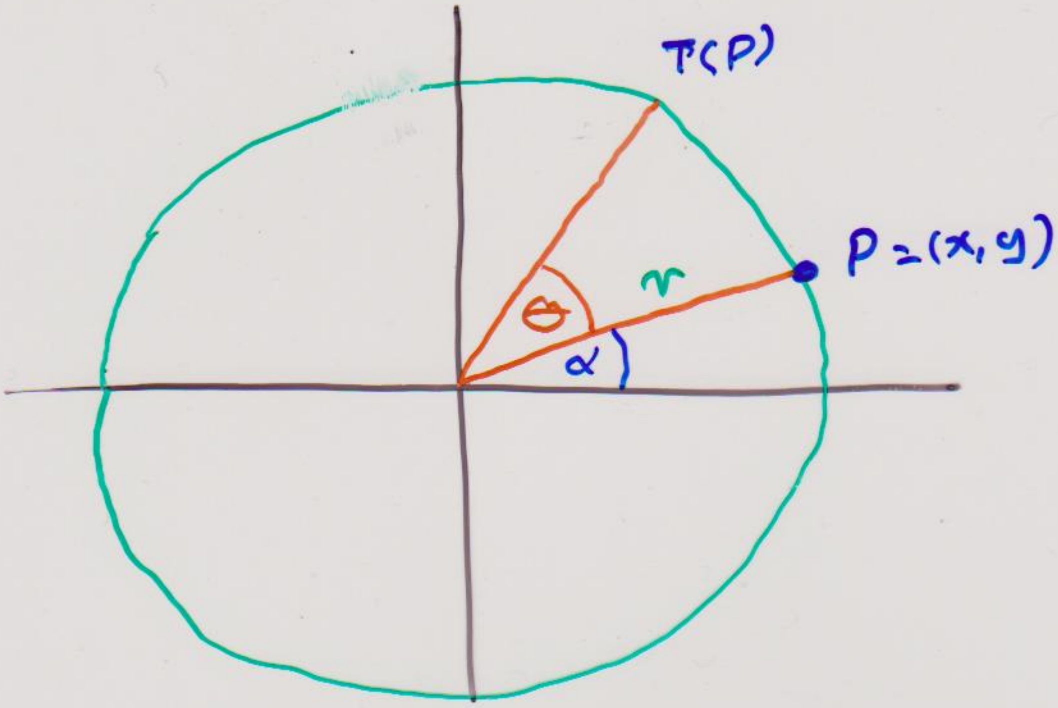
<u>Remark</u> The <u>trace</u> of a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the number $a+d$.

Theorem Let $S: \mathbb{R}^2 \to \mathbb{R}^2$, $T: \mathbb{R}^2 \to \mathbb{R}^2$ be linear transformations represented by matrices A, B. Then the linear transformation

$$T \circ S : \mathbb{R}^2 \to \mathbb{R}^2, \quad v \mapsto T(S(v))$$

is represented by the matrix BA.

Consider a rotation of the plane through an angle θ about the origin. What matrix represents this transformation?



if $P = (x, y) = (r\cos\alpha, r\sin\alpha)$

then

$$T(P) = (r\cos(\alpha+\theta), r\sin(\alpha+\theta))$$

$$= r(\cos(\alpha+\theta), \sin(\alpha+\theta))$$

$$= r(\cos\alpha\cos\theta - \sin\alpha\sin\theta, \sin\alpha\cos\theta + \cos\alpha\sin\theta)$$

$$= (x\cos\theta - y\sin\theta, y\cos\theta + x\sin\theta)$$

So

$$\tau(P) = \begin{pmatrix} x\cos\theta - y\sin\theta \\ \\ y\cos\theta + x\sin\theta \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \\ \sin\theta & \cos\theta \end{pmatrix}}_{\text{matrix of anticlockwise rotation through angle } \theta \text{ about origin.}} \begin{pmatrix} x \\ \\ y \end{pmatrix}$$