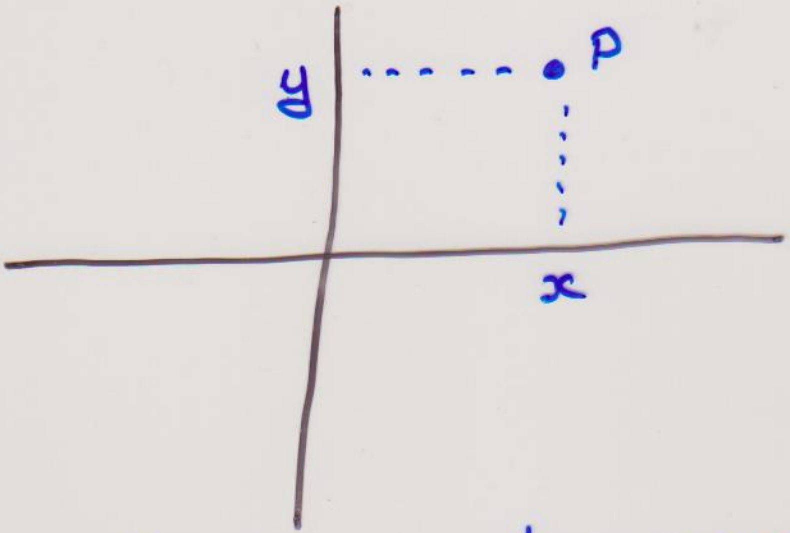
Linear transformations of the plane

$\mathbb{R}^2$ is the x-y-plane



Any point P can be represented by a pair of numbers

$$P = (x, y)$$

A transformation of the plane is just a function

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

which sends each point P to some point $T(P)$.

We can add two points

$$P = (x,y) \, , \quad Q = (x', y')$$

using the matrix addition

$$P+Q = (x+x', y+y') \, .$$

We can multiply a point $P=(x,y)$ by a scalar $\lambda \in \mathbb{R}$ using

$$\lambda P = (\lambda x, \lambda y) \, .$$

<u>Definition</u> A transformation

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

is said to be <u>linear</u> if

i) $T(P+Q) = T(P) + T(Q)$

ii) and $T(\lambda P) = \lambda T(P)$

for all $P, Q \in \mathbb{R}^2$ and all $\lambda \in \mathbb{R}$.

<u>Example</u> Consider the transformation

$$T : \mathbb{R}^2 \to \mathbb{R}^2, \ (x, y) \mapsto (x+2y, \ 3x+4y).$$

So

$T(-1, 2) = (3, 5)$

Is T linear ?

Consider $P = (x, y)$, $Q = (x', y')$

$$T(P+Q) = T(x+x', \ y+y')$$

$$= \left(x+x'+2(y+y'), \ 3(x+x')+4(y+y') \right)$$

$$= (x+2y+x'+2y', \ 3x+4y+3x'+4y')$$

$$= (x+2y, \ 3x+4y) + (x'+2y', \ 3x'+4y')$$

$$= T(x, y) + T(x', y')$$

$$= T(P) + T(Q).$$

Now :

$$T(\lambda P) = T(\lambda(x, y))$$

$$= T(\lambda x, \lambda y)$$

$$= (\lambda x + 2\lambda y, 3\lambda x + 4\lambda y)$$

$$= \lambda(x + 2y, 3x + 4y)$$

$$= \lambda T(P).$$

As this holds for all λ and all P, Q we see that T is indeed linear.

<u>Example</u> Consider

$$T : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto (x^2, y^2)$$

well, for $P = (3, 7)$ and $\lambda = 5$ we have :

$T(\lambda P) = T(15, 35)$

$= (225, 1225)$

$\lambda T(P) = 5\, T(3, 7)$

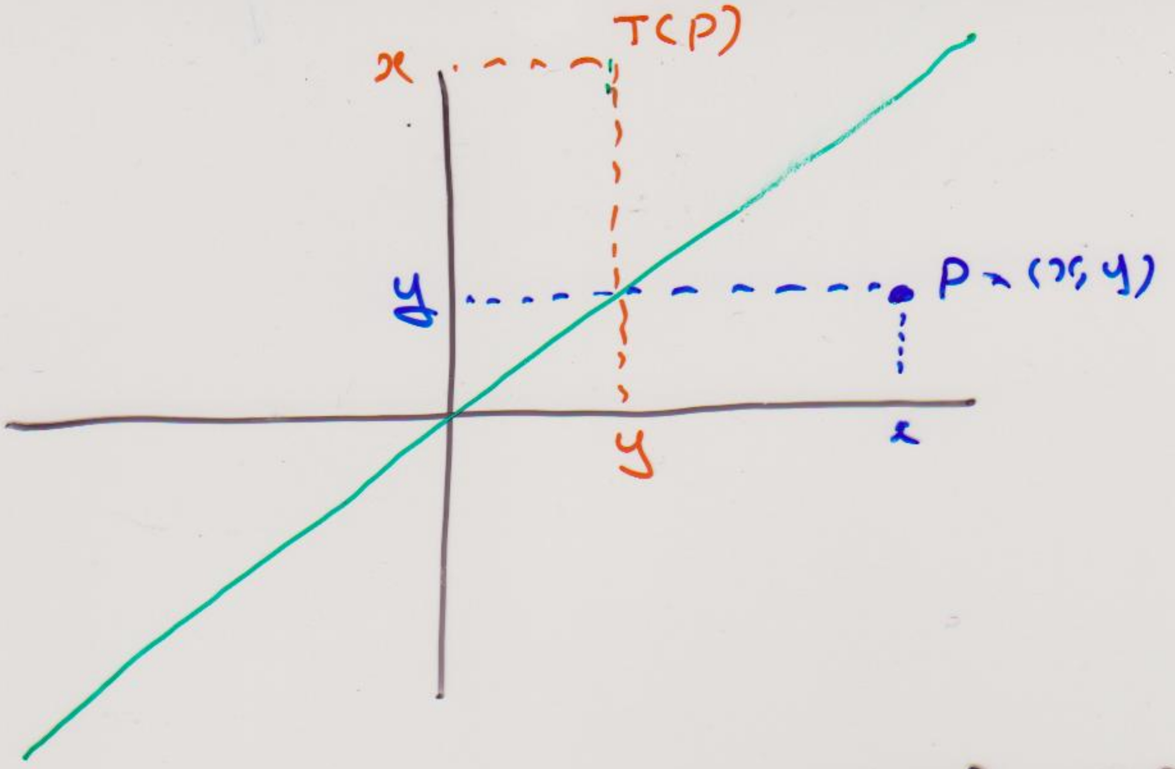
$= 5\,(9, 49)$

$= (45, 245)$.

Since $T(\lambda P) \neq \lambda T(P)$ this function is not linear.

<u>Example</u> Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation obtained by reflecting in the line $y = x$.

is this linear?



So a formula for $T: \mathbb{R}^2 \to \mathbb{R}^2$

is

$$T(x, y) \longmapsto (y, x)$$

is this linear?

for $P = (x, y)$, $Q = (x', y')$

$$\begin{aligned} T(P+Q) &= T(x+x', y+y') \\ &= (y+y', x+x') \\ &= (y, x) + (y', x') \\ &= T(x, y) + T(x', y') \\ &= P(Q) + T(Q). \end{aligned}$$

$$T(\lambda P) = T(\lambda x, \lambda y) = (\lambda y, \lambda x) = \lambda T(x, y)$$

So yes, the reflection is linear.

Example 2 Suppose we rotate the plane anticlockwise through an angle θ about the origin. Is this transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ linear?