Differentiation

Recall, given a function $y = f(x)$, we define the derivative $f'(x)$ as

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

Example $y = x^2$, or $f(x) = x^2$.

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{(\cancel{x^2} + 2hx + h^2) - \cancel{x^2}}{h}$$

$$= \lim_{h \to 0} \frac{2\cancel{h}x + h^{\cancel{2}}}{\cancel{h}}$$

$$= \lim_{h \to 0} 2x + h$$

$$= 2x$$

Derivatives of some basic functions

we often write $\frac{dy}{dx}$ instead of $f'(x)$.

- $\frac{d}{dx} x^n = n x^{n-1}$ (for any number n)
- $\frac{d}{dx} \sin(x) = \cos(x)$
- $\frac{d}{dx} \cos(x) = -\sin(x)$
- $\frac{d}{dx} \tan(x) = \sec^2 x$
- $e = 2.71 \cdots$

$$\frac{d}{dx} e^x = e^x$$

Rules for differentiation

- $$\boxed{\frac{d}{dx}\left(f(x) + g(x) \right) = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)}$$

Sum Rule

Example

$$\frac{d}{dx}\left(x^{3/2} + \sin(x) \right)$$

$$= \frac{d}{dx} x^{\frac{3}{2}} + \frac{d}{dx} \sin(x)$$

$$= \frac{3}{2} x^{\frac{1}{2}} + \cos(x) .$$

- for a constant k

$$\boxed{\frac{d}{dx}\left(k f(x) \right) = k \frac{d}{dx} f(x)}$$

scalar Rule

Example

$$\frac{d}{dx} 3e^{x} = 3 \frac{d}{dx} e^{x} = 3e^{x}$$

$$\boxed{\frac{d}{dx}\left(f(x)\,g(x)\right) = \left(\frac{d}{dx}f(x)\right)g(x) + f(x)\left(\frac{d}{dx}g(x)\right)}$$

Product Rule

Example

$$y = x^2 \sin(x)$$

$$\frac{d}{dx}y = \left(\frac{d}{dx}x^2\right)\sin(x) + x^2\left(\frac{d}{dx}\sin x\right)$$

$$\frac{dy}{dx} = 2x\sin(x) + x^2\cos(x).$$

Example

$$y = (x^2+1)(x^3+2)$$

$$\frac{dy}{dx} = 2x(x^3+2) + (x^2+1)(3x^2)$$

$$= 2x^4 + 4x + 3x^4 + 3x^2$$

Chain Rule

Given functions

$f(x)$ and $g(x)$

we can consider the function

$$y = g(f(x))$$

$$\boxed{\frac{dy}{dx} = g'(f(x)) \cdot f'(x)}$$

CHAIN RULE

Example

$$y = \sin(x^2)$$

$g(x) = \sin(x)$, $f(x) = x^2$

$$\frac{dy}{dx} = \cos(x^2)\,2x = 2x\cos(x^2)$$

Example

$$y = (x^2 - x + 1)^7$$

$g(x) = x^7$

$f(x) = x^2 - x + 1$

$$\frac{dy}{dx} = 7(x^2-x+1)^6(2x-1)$$

Example

$$y = \sqrt{x^2+1}$$

$g(x) = \sqrt{x}$ $\qquad$ $f(x) = x^2+1$

$g(x) = x^{\frac{1}{2}}$

$g'(x) = \frac{1}{2}x^{-\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{1}{\cancel{2}}(x^2+1)^{-\frac{1}{2}}(\cancel{2}x)$$

$$= \frac{x}{\sqrt{x^2+1}}$$

Quotient Rule

$$y = \frac{f(x)}{g(x)}$$

$$\frac{dy}{dx} = \frac{d}{dx} f(x) \left(g(x)\right)^{-1}$$

$$= f'(x) \cdot \left(g(x)\right)^{-1} + f(x) \cdot \overbrace{\frac{d}{dx}\left(g(x)\right)^{-1}}^{\text{chain rule}}$$

$$= \frac{f'(x)}{g(x)} + f(x)\left(\frac{-1}{g(x)^2}\right) g'(x)$$

$$= \boxed{\frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}}$$

Quotient Rule

Examp

$$y = \frac{x^2 + x - 2}{x^3 + 6}$$

$$= \frac{(x^3+6)(2x+1) - (x^2+x-2)(3x^2)}{(x^3+6)^2}$$