MA140 Christmas Test

- Last Tuesday lecture
- Counts same as one homework
- Two questions, taken verbatim, from the first 4 questions on the Summer 2011 MA140 exam.

An equation involving a derivative

$$\frac{dy}{dt} = ky \qquad (*)$$

Let's try to solve this equation.

1) Are there any solutions?

For instance

$$y = e^{kt}$$

is a solution since

$$\underline{\underline{\frac{dy}{dt}}} = \frac{d}{dt}e^{kt} = ke^{kt} = k\underline{\underline{y}}.$$

Also,

$$y = 3e^{kt}$$

is a solution since

$$\frac{dy}{dt} = \frac{d}{dt}\left(3e^{kt}\right) = 3ke^{kt} = k\,3e^{kt} = ky.$$

Are there any other type of solutions to $(*)$?

Suppose $y = y(t)$ and $z = z(t)$ are both solutions to $(*)$.

Thus

$$\frac{d}{dt}\left(\frac{y}{z}\right) = \frac{z'y - y'z}{z^2}$$

$$= \frac{kzy - kyz}{z^2} = 0.$$

So $\frac{y}{z}$ is a constant,

Say $\frac{y}{z} = A$ a constant.

Hence

$$y = Az.$$

Conclusion: The only solutions to the differential equation (*) are

$$(\dagger) \quad \boxed{y = A e^{kt} \qquad A \text{ constant.}}$$

<u>Example</u>

A cup of coffee in a room at $20°C$ cools from $80°C$ to $50°C$ in five minutes.
How long will it take to cool to $40°C$?

<u>Soln</u>

<u>Newton</u>: A hot object cools at a rate proportional to the excess of its temperature above room temperature.

$y(t)$ = temperature of coffee at time t

$y(0) = 80$

$y(5) = 50$

Question: for what t do we have $y(t) = 40$.

Newton:

$$\frac{dy}{dt} = k(y-20)$$

Consider $z = y - 20$

$z(0) = 60$

$z(5) = 30$

$$\frac{dz}{dt} = \frac{d}{dt}(y-20) = \frac{dy}{dt} \overset{\text{Newton}}{=} k(y-20) = kz$$

and

$$\boxed{\frac{dz}{dt} = kz}$$

From (+) we know that z must be of form

$$z = A e^{kt} \quad , \quad A, k \text{ constant.}$$

Let's find A, k.

$$60 = z(0) = A e^{k0} = A e^{0} = A$$

$$\boxed{A = 60}$$

So

$$z = 60 e^{kt} .$$

$$30 = z(5) = 60 e^{5k}$$

$$\boxed{e^{5k} = \frac{1}{2}}$$

Question: for what t do we have $z(t) = 20$?

Need to solve for t:

$$20 = z(t) = 60 e^{kt}$$

or

$$\frac{1}{3} = e^{kt}$$

or

$$\frac{1}{3} = \left(e^{5k}\right)^{t/5}$$

or

$$\frac{1}{3} = \left(\frac{1}{2}\right)^{\frac{t}{5}}$$

or

$$\ln\left(\frac{1}{3}\right) = \ln\left(\left(\frac{1}{2}\right)^{\frac{t}{5}}\right) = \frac{t}{5}\ln\left(\frac{1}{2}\right)$$

So

$$t = 5\,\frac{\ln\left(\frac{1}{3}\right)}{\ln\left(\frac{1}{2}\right)} \text{ minutes.}$$