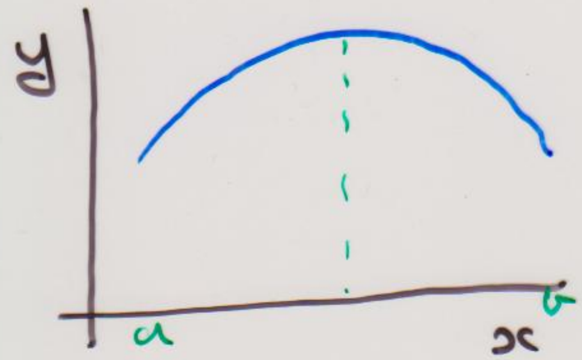
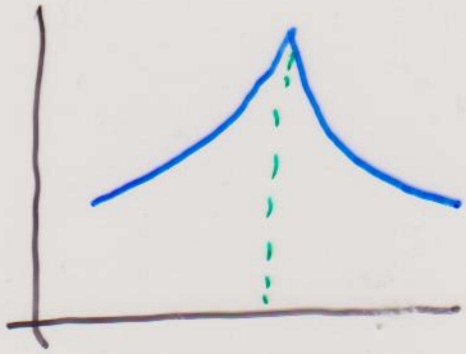
More on curve sketching

Recall

$y = f(x)$

local maxima look like:





y' $+ + + + 0 - - - -$

$+ + + \mid - - -$

y' does not exist at max.

Defn If y'' is $\begin{cases} \text{positive} \\ \text{negative} \end{cases}$ on an interval $a \le x \le b$ then we say that the function y is $\begin{cases} \text{concave up} \\ \text{concave down} \end{cases}$ on the interval.

__Defn__ A point x at which $y = f(x)$ changes from being concave down to concave up, or from concave up to concave down, is called a __point of inflection__.

y'' $+ + + + + + + + 0 - - - - - - -$

__Problem__ Sketch the graph of $y = x e^{-\frac{x^2}{2}}$, indicating any local maxima, minima and points of inflection.

Sol 4

Note: y is defined for all values of x.

$$\lim_{x \to \infty} x e^{-\frac{x^2}{2}} = \lim_{x \to 0} \frac{x}{e^{x^2/2}} = 0$$

$$\lim_{x \to -\infty} x e^{-\frac{x^2}{2}} = 0$$

So the line $y = 0$ is a horizontal asymptote.

$$y = x e^{-\frac{x^2}{2}}$$

$$y' = x \frac{d}{dx}\left(e^{-x^2/2}\right) + \frac{d}{dx}(x)\, e^{-\frac{x^2}{2}}$$

$$= x\, e^{-\frac{x^2}{2}} \frac{d}{dx}\left(-\frac{x^2}{2}\right) + e^{-x^2/2}$$

$$= -x^2 e^{-\frac{x^2}{2}} + e^{-\frac{x^2}{2}}$$

$$\boxed{y' = (1 - x^2)\, e^{-\frac{x^2}{2}}}$$

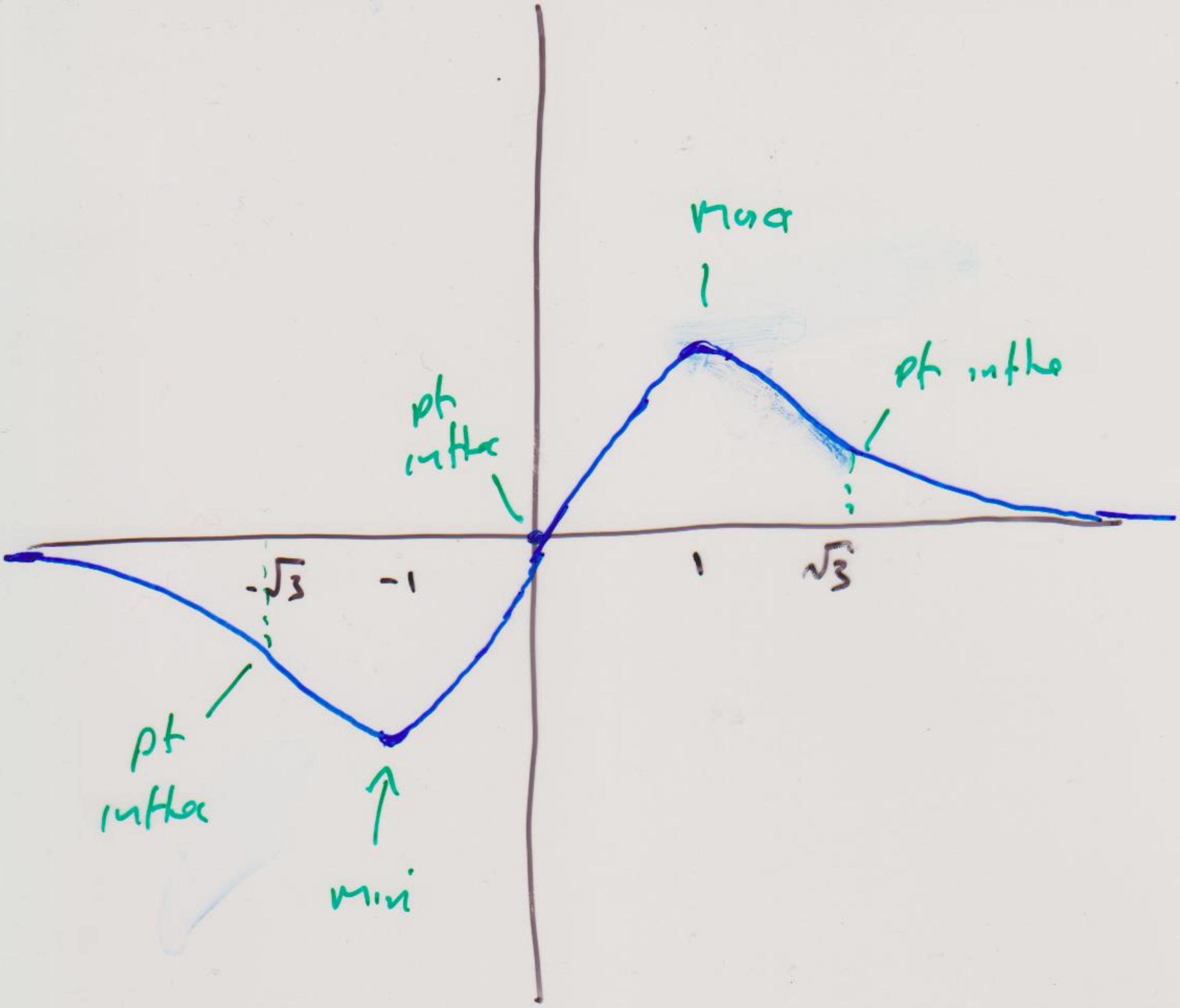
$$y'' = \left\{\frac{d}{dx}(1-x^2)\right\} e^{-\frac{x^2}{2}} + (1-x^2)\,\frac{d}{dx} e^{-\frac{x^2}{2}}$$

$$= -2x\, e^{-\frac{x^2}{2}} + -x(1-x^2)\, e^{-\frac{x^2}{2}}$$

$$= (x^3 - 3x)\, e^{-\frac{x^2}{2}}$$

$$\boxed{y'' = x(x^2-3)\, e^{-\frac{x^2}{2}}}$$

x		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$	
y'	$-$	$-$	$-$	0	$+$	$+$	$+$	0	$-$	$-$	$-$
y''	$-$	0	$+$	$+$	$+$	0	$-$	$-$	$-$	0	$+$
y	decreasing				increasing				decreasing		
		pt of inflection		min		pt inflec		max		pt inflec	


max
pt inflec
pt inflec
pt inflec
mini
$-\sqrt{3}$
-1
1
$\sqrt{3}$