Applications of Derivatives

First Application: derivatives can be used to calculate limits.

Recall:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

So for very small h we have approximately

$$f'(x) \simeq \frac{f(x+h) - f(x)}{h}$$

or

$$h f'(x) \simeq f(x+h) - f(x)$$

or

$$\boxed{f(x) \simeq f(x+h) - h f'(x)}$$

Suppose $f(x)$ and $g(x)$ are continuous functions, and that for some c we have $f(c) = 0$ and $g(c) = 0$.

now

$$\lim_{x \to c} \frac{f(x)}{g(x)} \simeq \lim_{x \to c} \frac{f(x+h) - hf'(x)}{g(x+h) - hg'(x)}$$

when h is small.

now $f(c+h) \simeq 0$ & $g(c+h) \simeq 0$

So

$$\lim_{x \to c} \frac{f(x)}{g(x)} \simeq \lim_{x \to c} \frac{-hf'(x)}{-hg'(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

<u>L'Hôpital's</u> <u>Theorem</u>

if $f(x)$ and $g(x)$ are continuous functions with $f(c) = g(c) = 0$,

then

$$\boxed{\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}}.$$

<u>Example</u> Evaluate

$$l = \lim_{x \to \frac{\pi}{2}} \tan(x) - \sec(x)$$

<u>Sol^n</u>

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

$$\sec(x) = \frac{1}{\cos(x)}$$

Now

$$l = \lim_{x \to \frac{\pi}{2}} \frac{\sin(x)}{\cos(x)} - \frac{1}{\cos(x)}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\sin(x) - 1}{\cos(x)}$$

$$\underset{\text{L'Hôpital}}{=} \lim_{x \to \frac{\pi}{2}} \frac{\cos(x)}{-\sin(x)} = \frac{0}{-1} = 0.$$

Example Evaluate

$$l = \lim_{x \to 0} \frac{\sin(x)\cos(x) - x}{x^3}$$

Sol^n

$$l \overset{\text{L'Hôpital}}{=} \lim_{x \to 0} \frac{\cos^2(x) - \sin^2(x) - 1}{3x^2}$$

$$\overset{\text{L'Hôpital}}{=} \lim_{x \to 0} \frac{-2\cos(x)\sin(x) - 2\sin(x)\cos(x)}{6x}$$

$$= \lim_{x \to 0} \frac{-4\cos(x)\sin(x)}{6x}$$

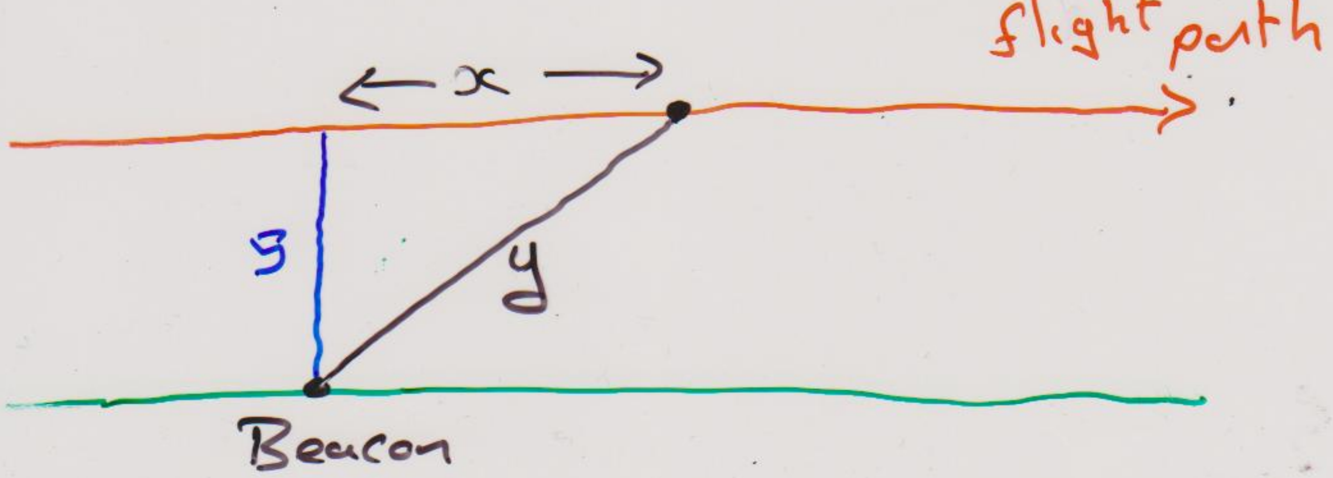
$$\overset{\text{L'Hôpital}}{=} \lim_{x \to 0} \frac{4\sin^2 x - 4\cos^2 x}{6}$$

$$= -\frac{4}{6} = -\frac{2}{3}.$$

Second Application: The derivative can be thought of as a rate of change.

Problem An aircraft is flying horizontally at a speed of 600 km/h. How fast is the distance between a radio beacon and the aircraft increasing 1 minute after the plane passes 5 km directly above the beacon?

Sol^n



We need to find $\frac{dy}{dt}$ when

$t=1$.

Well

when $t=1$, $x=10$.

and

$$5^2+x^2=y^2.$$

Chain rule

$$\frac{dy}{dt}=\frac{dy}{dx}\frac{dx}{dt} \quad (*)$$

$$y=\sqrt{25+x^2}=(25+x^2)^{\frac{1}{2}}$$

$$\frac{dy}{dx}=\frac{1}{2}(25+x^2)^{-\frac{1}{2}}2x=\frac{x}{\sqrt{25+x^2}}$$

at $t=1$, $\frac{dx}{dt}=\cancel{600}\ 10$

So from (*)

$$\frac{dy}{dt} = \frac{x}{\sqrt{25+x^2}} \quad \cancel{600}\ 10$$

when $t = 1$, $x = 10$ and

$$\frac{dy}{dt} = \frac{10}{\sqrt{125}}10 = \frac{100}{5\sqrt{5}}$$

$$= \frac{20}{\sqrt{5}} \text{ km/m}$$