OLLSCOIL NA hÉIREANN, GAILLIMH
NATIONAL UNIVERSITY OF IRELAND, GALWAY

SUMMER EXAMINATIONS, 2009

FIRST ENGINEERING & INFORMATION TECHNOLOGY EXAMINATION

MATHEMATICS [MA150]

MA151 — *Calculus*

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Time allowed: *Three* hours.
Answer six questions.

1. (a) Find the equation of the tangent to the curve $y = \sqrt{x}$ at $x = 4$.

(b) Show that

$$2 + 8x - 2x^3 = 0$$

has three real solutions. (Hint: consider sign changes.)

(c) For what value of k is the following function $g(x)$ continuous at all points? And, for this value of k, is $g(x)$ differentiable at all points?

$$g(x) = \left\{ \begin{array}{ll} kx & \text{if } x < 2 \\ x^2 + k & \text{if } x \geq 2 \end{array} \right\}$$

p.t.o.

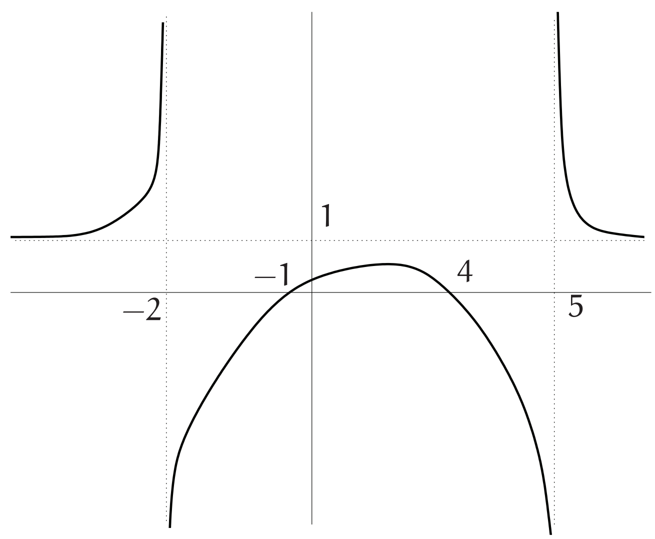
2. (a) Evaluate the following.

(i) $\lim_{\theta \to 0} \frac{\cos(\theta) - 1}{\theta}$

(ii) $\lim_{x \to \sqrt{3}} \frac{x - \sqrt{3}}{x^2 - 3}$

(iii) $\lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$ where $f(x) = \sqrt{x}$

(b) Find polynomials $p(x)$ and $q(x)$ such that a sketch of the function $f(x) = p(x)/q(x)$ looks as follows (with vertical asymptotes $x = -2$, $x = 5$, horizontal asymptote $y = 1$ and $f(-1) = f(4) = 0$).



3. (a) Differentiate the following.

(i) $y = \sin(x^3)$

(ii) $y = \frac{\sin(x)}{\sqrt{x}}$

(iii) $y = \frac{(x+1)\sqrt{x+2}}{(x+3)\sqrt{x+4}}$ (Hint: logarithmic differentiation)

(b) Suppose an open box is made from a square sheet of metal of side 1m by cutting equal squares from the corners and folding up the sides. What is the largest possible volume for such a box?

4. (a) An aircraft is flying horizontally at a speed of 600 km/h. How fast is the distance between the aircraft and a radio beacon increasing 1 minute after the aircraft passes 5km directly above the beacon?

(b) A function $y(t)$ satisfies the differential equation

$$\frac{dy}{dt} = k(y - 20)$$

for some constant k. Suppose that $y(0) = 80$ and $y(5) = 50$. Determine the value of t for which $y(t) = 40$.

5. (a) State the Fundamental Theorem of Calculus, and use it to show that

$$\frac{d}{dx}\int_x^{\tan x} \frac{1}{1+t^2}\, dt = \frac{x^2}{1+x^2}.$$

(b) Let $f(x) = \sin x$. Estimate $\int_0^{\pi/2} f(x)\, dx$ by calculating the lower Riemann sums $L(f, P_n)$ and the upper Riemann sums $U(f, P_n)$ for $n = 3$ with respect to a partition P_n of $[0, \frac{\pi}{2}]$ into n subintervals of equal length. Verify that

$$L(f, P_3) \leq 1 \leq U(f, P_3)$$

and explain why $L(f, P_n) \leq 1 \leq U(f, P_n)$ for all $n > 0$.

6. (a) Find the following three integrals.

(i) $\int \sec^4 x\, dx$, (ii) $\int_0^{\sqrt{3}} \frac{x}{x^4+9}\, dx$, (iii) $\int \frac{x^2+2x-1}{(x^2+1)(x+1)}\, dx$.

(b) For $n \geq 0$, define $I_n = \int_0^{\pi} x^n \sin x\, dx$. Use integration by parts to show that for $n \geq 2$

$$I_n = \pi^n - n(n-1)I_{n-2}.$$

Compute I_0 and hence evaluate $\int_0^{\pi} x^4 \sin x\, dx$.

7. (a) Find the area A of the region bounded by the curves $y = 2 - x^2$ and $y = x + 4$ and the vertical lines $x = 2$ and $x = -2$.

(b) Find the length L of the curve parametrized by

$$x = \sqrt{5}\sin(2t) - 2, \qquad y = \sqrt{5}\cos(2t) - \sqrt{3},$$

from $t = 0$ to $t = \frac{1}{4}\pi$.

(c) Sketch the region bounded by the curves $y = x^{3/2}$, $y = 8$ and $x = 0$ and find the volume V of the solid generated by revolving the region about the y-axis.

8. Solve the following differential equations:

(a) $\dfrac{dy}{dx} + xy = x^3$, (b) $\dfrac{dy}{dx} = \dfrac{x+y}{x-y}$, (c) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} - 3y = 0$.