Omnibus Arts MH1020

Mathematical Studies
&
Mathematics

BA Connect MH100

Mathematical Studies
&
Mathematics

1 MH1020 Mathematical Studies

MH1020 Mathematical Studies is a one-year introduction to University mathematics which emphasizes calculations and applications for a range of topics in algebra and calculus. It is a natural continuation of school level Project Maths and is suitable for students with an OB3 or better in Leaving Certificate Mathematics. Students who score over 50% in MH1020 Mathematical Studies are encouraged to go on to study for an Honours BA degree in Mathematical Studies. This degree provides students with a highly employable numerate training. It also meets the Teaching Council's subject requirements for entering the Mathematics teaching profession. Recent graduates have found employment in areas such as finance, administration, teaching and the civil service. Others have gone on to the 1-year Higher Diploma in Mathematics before starting their professional careers.

MH1020 Mathematical Studies consists of four modules.

- MA133 Calculus and Algebra:

  Semester I, four lectures per week.

  Delivered by Dr John Burns & Dr James Cruickshank in 2016/17.

  Examined by a two-hour exam at the end of the semester.

- MA135 Calculus and Algebra:

  Semester II, four lectures per week.

  Delivered by Prof Graham Ellis in 2016/17.

  Examined by a two-hour exam at the end of the semester.

- MA131 Mathematical Skills:

  Semesters I & II, one tutorial per week.

  Examined by twelve online homeworks.

- MA1222 Mathematical Cultures:

  Semester II, one lecture per week.

  Delivered by Prof Dane Flannery in 2016/17.

  Examined by essay and a two-hour exam at the end of the semester.

The Director of the Mathematical Studies BA programme is Dr John Burns.

1.1 MH100 Mathematical Studies for BA Connect

Students in the BA Connect programme take MA131, MA133 and MA135 only.

2 MH1020 Mathematics

MH1020 Mathematics is a one-year introduction to University mathematics which emphasizes conceptual and theoretical aspects of a range of topics in algebra and calculus. It is a natural continuation of school level Project Maths and is suitable for students with an OA2 or HC3 or better in Leaving Certificate Mathematics. Students who score over 50% in MH1020 Mathematics are encouraged to go on to study for an Honours BA degree in Mathematics. This degree provides students with a highly employable training in advanced mathematics. It more than meets the Teaching Council's subject requirements for entering the Mathematics teaching profession. Recent graduates have found employment in areas such as finance, administration, teaching and the civil service. Others have gone on to the 1-year MA in Mathematics and then on to a PhD degree in mathematics before starting their professional careers.

MH1020 Mathematics consists of four modules.

- MA185 Calculus and Algebra:

  Semester I, four lectures per week.

  Delivered by Dr Dieter Degrijse & Prof Graham Ellis in 2016/17.

  Examined by a two-hour exam at the end of the semester.

- MA186 Calculus and Algebra:

  Semester II, four lectures per week.

  Delivered by Prof Götz Pfeiffer and Dr Rachel Quinlan in 2016/17.

  Examined by a two-hour exam at the end of the semester.

- MA187 Mathematical Skills:

  Semesters I & II, one tutorial per week.

  Examined by twelve online homeworks.

- MA1222 Mathematical Cultures:

  Semester II, one lecture per week.

  Delivered by Prof Dane Flannery in 2016/17.

  Examined by essay and a two-hour exam at the end of the semester.

The Director of the Mathematics BA programme is Dr John Burns.

2.1 MH100 Mathematics for BA Connect

Students in the BA Connect programme take MA185, MA186 and MA187 only.

3 Choosing between Mathematical Studies and Mathematics

The following table lists a few of the differences between the Mathematics BA and the Mathematical Studies BA.

	Mathematical Studies	**Mathematics**
Recommended minimum Leaving Cert Maths grade:	OB3	OA2 or HC3
Leads on to:	HDip in Mathematics	MA in Mathematics
Emphasizes:	mathematical calculations; mathematical applications; cultural context of mathematics	mathematical concepts; mathematical theory; cultural context of mathematics
Unique features include:	Computing module in 3rd Year	Group Theory module and Topology Module in 3rd Year
Typical number of students in 3rd Year:	20-35	5-10
Requirements for entry to 2nd Year:	MA131, MA1222, MA133/MA185, MA135/MA186	MA131/MA187, MA1222, MA185, MA186

4 Changing between Mathematical Studies and Mathematics

Students who choose the MA185 Mathematics module can change to the Mathematical Studies MA135 module at any point during the first three weeks of Semester I; after that point they must stay with MA185 until the end of the semester. Students who complete MA185 can choose between the MA186 Mathematics module and the MA135 Mathematical Studies module in Semester II. Students who complete MA185 and MA186 can choose between the Mathematics BA programme and the Mathematical studies BA programme in Second Year.

Students who choose the MA135 Mathematical Studies module are not allowed to move to the MA185 Mathematics module nor to take the MA186 module. The MA133 and MA135 modules lead to Mathematical Studies in Second Year.

5 Algebra & Calculus MA133 and MA131-I

The MA133 module covers six topics in algebra and calculus. These topics are represented by the six sections of problems given in Section 10 below. The module is examined by a 2-hour exam at the end of Semester I consisting of six questions, one per topic.

Semester: I

Algebra Lecturer: Dr John Burns.

Algebra Text: *Algebra & Geometry: An introduction to University Mathematics* by Mark V. Lawson.

Algebra Lectures: Wednesday 10am & Thursday 10am.

Calculus Lecturer: Dr James Cruickshank.

Calculus Text: *Calculus. Early Transcendentals* by James Stewart.

Calculus lectures: Monday 1pm & Tuesday 10am.

Tutorial: Monday 12pm.

The continuous assessment for MA133 consists of six online homeworks. These six homeworks count for 50% of the Mathematical Skills module MA131.

Online homework	Topic	Deadline
First homework	Functions and graphs & Matrix algebra	5pm Friday, Week 3
Second homework	Functions and graphs & Matrix algebra	5pm Friday, Week 5
Third homework	Differentiation & Eigenvalues of 2×2 matrices	5pm Friday, Week 7
Fourth homework	Differentiation & Eigenvalues of 2×2 matrices	5pm Friday, Week 9
Fifth homework	Maxima, minima, related rates & Number theory	5pm Friday, Week 11
Sixth homework	Maxima, minima, related rates & Number theory	5pm Friday, Week 12

6 Algebra & Calculus MA185 and MA187-I

The MA185 module covers six topics in algebra and calculus. These topics are represented by the six problems of the sample exam paper in Section 10 below. The module is examined by a 2-hour exam at the end of Semester I consisting of six questions, one per topic.

Semester: I

Algebra Lecturer: Prof Graham Ellis.

Algebra Text: *Algebra & Geometry: An introduction to University Mathematics* by Mark V. Lawson.

Algebra Lectures: Wednesday 10am & Thursday 10am.

Calculus Lecturer: Dr Dieter Degrijse.

Calculus Text: *Calculus. Early Transcendentals* by James Stewart.

Calculus lectures: Monday 1pm & Tuesday 10am.

Workshop: Monday 12pm.

The continuous assessment for MA185 consists of six online homeworks. These six homeworks count for 50% of the Mathematical Skills module MA187.

Online homework	Topic	Deadline
First homework	Functions and continuity & Number theory and cryptography	5pm Friday, Week 3
Second homework	Functions and continuity & Number theory and cryptography	5pm Friday, Week 5
Third homework	Differentiation & Matrix arithmetic	5pm Friday, Week 7
Fourth homework	Differentiation & Matrix arithmetic	5pm Friday, Week 9
Fifth homework	Differential equations & Eigenvectors and recurrence	5pm Friday, Week 11
Sixth homework	Differential equations & Eigenvectors and recurrence	5pm Friday, Week 12

7 Algebra & Calculus MA135 and MA131-II

The MA135 module covers a further six topics in algebra and calculus. The module is examined by a 2-hour exam at the end of Semester II consisting of six questions, one per topic.

Semester: II

Algebra Lecturer: Prof Graham Ellis.

Algebra Text: *Algebra & Geometry: An introduction to University Mathematics* by Mark V. Lawson.

Algebra Lectures: Wednesday 10am & Thursday 10am

Calculus Lecturer: Prof Graham Ellis.

Calculus Text: *Calculus. Early Transcendentals* by James Stewart.

Calculus lectures: Monday 1pm & Tuesday 10am.

Tutorial: Monday 12pm

The continuous assessment for MA135 consists of six online homeworks. These six homeworks count for the remaining 50% of the Mathematical Skills module MA131.

Online homework	Topic	Deadline
First homework	Integration & Logic	5pm Friday, Week 3
Second homework	Integration & Logic	5pm Friday, Week 5
Third homework	Techniques of integration & Systems of equations	5pm Friday, Week 7
Fourth homework	Techniques of integration & Systems of equations	5pm Friday, Week 9
Fifth homework	Differential equations & Complex numbers	5pm Friday, Week 11
Sixth homework	Differential equations & Complex numbers	5pm Friday, Week 12

8 Algebra & Calculus MA186 and MA187-II

The MA186 module covers a further six topics in algebra and calculus. The module is examined by a 2-hour exam at the end of Semester II consisting of six questions, one per topic.

Semester: II

Algebra Lecturer: Prof Götz Pfeiffer.

Algebra Text: *Algebra & Geometry: An introduction to University Mathematics* by Mark V. Lawson.

Algebra Lectures: Wednesday 10am & Thursday 10am.

Calculus Lecturer: Dr Rachel Quinlan.

Calculus Text: *Calculus. Early Transcendentals* by James Stewart.

Calculus lectures: Monday 1pm & Tuesday 10am.

Tutorial: Monday 12pm

The continuous assessment for MA186 consists of six online homeworks. These six homeworks count for the remaining 50% of the Mathematical Skills module MA187.

Online homework	Topic	Deadline
First homework	Integration & Logic	5pm Friday, Week 3
Second homework	Integration & Logic	5pm Friday, Week 5
Third homework	Cardinality and the real number system& Polynomial arithmetic and permutations	5pm Friday, Week 7
Fourth homework	Cardinality and the real number system& Polynomial arithmetic and permutations	5pm Friday, Week 9
Fifth homework	Differential equations & $n \times n$ matrix algebra	5pm Friday, Week 11
Sixth homework	Differential equations & $n \times n$ matrix algebra	5pm Friday, Week 12

9 Mathematical Cultures MA1222

The MA1222 module covers topics in the History of Mathematics. The module is assessed by essay and a 2-hour exam ath the end of Semester II.

Semester: II

Mathematical Cultures Lecturer: Prof Dane Flannery.

Mathematical Cultures Text: *TBD*.

Mathematical Cultures Lectures: Friday 3pm.

10 MA133 Problems for 2016/17

10.1 Functions and graphs

1. [Linear functions]
   As dry air moves upwards it expands and cools. The temperature T of the air is a linear function of the height h. The ground temperature is $20°$C and the temperature at a height of 1km is $10°$C.

   (a) Express T (in $°$C) as a function of h (in km).

   (b) What is the temperature at a height of 2.5km?

   (c) At what height will the air temperature be $-15°$C?

   (d) Sketch the graph of the function $T(h)$. What does the slope of the graph represent?

   (e) The expression for T clearly depends on location and time since ground temperature varies with these two factors. Two Met Eireann weather balloons pass above Galway on a given dry day and measure the air temperature to be $-5°$C at a height of 3km and $10°$C at a height of 1.5km. What is the temperature in Galway on that day?

   (*cf.* [Stewart], Sec. 1.2, Example 1)

2. [Polynomial functions]
   A short-term economic model assumes that a country's GDP G (in €100 billion) at time t (in years) can be expressed as a quadratic polynomial $G = at^2 + bt + c$.

   Initially G has a value of 20. At 6 months the value of G is 29 and at 1 year the value of G is 40.

   (a) Determine the values of the constants a, b, c.

   (b) Evaluate G for $t = 0, 1, 2, 3, 4, 5$ years.

   (c) Sketch the graph of $G(t)$ over the range $0 \leq t \leq 5$.

   (d) The *growth* over a time interval starting at time t_1 and ending at time t_2 is defined to be the number
   $$\frac{G(t_2) - G(t_1)}{G(t_1)} .$$
   Calculate the predicted growth for the interval from $t_1 = 1$ to $t_2 = 2$ years.

   (e) The EU defines a country to be *in recession* at time t if its growth is negative for the interval from $t_1 = t - 0.25$ to $t_2 = t$ years and also for the interval from $t_1 = t - 0.5$ to $t_2 = t - 0.25$ years. What does the economic model predict about recession at time $t = 30$ months?

3. [Rational functions]
   For each of the formulae

$$(i)\ f(x) = \frac{1}{1-x}\,, \quad (ii)\ f(x) = \frac{x}{1-x}\,, \quad (iii)\ f(x) = \frac{x-2}{1-x}$$

   (a) determine all those real numbers x for which $f(x)$ is not defined.
   (b) evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.
   (c) sketch the graph of $f(x)$.

   Then,

   (d) for $x = 0.8$, evaluate the sum $x + x^2 + x^3 + x^4 + x^5 + \cdots$ involving infinitely many powers of x.
   (e) determine the amount of money that needs to be deposited in a bank account in order to finance an annual payment of €1 in perpetuity, assuming that the bank account will forever pay fixed interest of $i = 25\%$ on deposits at the end of each year. (Let $x = 1/(1+i)$ and note that €x earns interest of €1 after 1 year, €x^2 earns interest of €1 after two years, and so forth.)

4. [Trigonometric functions]
   A lighthouse is located on a small island 3 km away from the nearest point P on a straight shoreline and its light beam makes one revolution every four minutes. The light beam passes point P at time $t = 0$ seconds.

   (a) Give a formula, valid for $0 \le t < 60$, for the distance D (in km) between the point P and the point where the light beam hits the shore at time t seconds.
   (b) Evaluate $D(t)$ for $t = 0, 15, 30, 45$ seconds.
   (c) Sketch the graph of $D(t)$ for the time interval $0 \le t \le 45$.
   (d) Determine the average speed of the light beam on the shoreline over the time interval $t = 0$ to $t = 15$ seconds.
   (e) Determine the average speed of the light beam on the shoreline over the time interval $t = 15$ to $t = 30$ seconds.

   (*cf.* [James], Sec. 3.5, Problem 38)

5. [Exponential function]
   An Essay on the Principle of Population, written by the Rev Thomas Robert Malthus and published in 1798, is one of the earliest and most influential books on population. It reasons that the size $P(t)$ of a population at a given time t is modelled by the equation

$$P(t) = Ae^{kt}$$

   where A and k are constants that depend on the population being modelled.

   Let us suppose that t is measured in years and that $t = 0$ corresponds to 1950.

(a) Use the fact that the world population was 2560 million in 1950 to determine the constant A for the world population.

(b) Use the fact that the world population was 3040 million in 1960 to determine the constant k for the world population.

(c) Use this *Malthusian model* to estimate the population of the world in 1993.

(*cf.* [Stewart], Sec. 3.8, Example 1)

6. [Limits of rational functions]
Aristotle had taught that heavy objects fall faster than lighter ones, in direct proportion to weight. Story has it that Galileo Galilei (1564-1642) dropped balls of the same material, but different masses, from the Leaning Tower of Pisa to demonstrate that their time of descent was independent of their mass. *Galileo's law* states that, ignoring air resistance, the distance $s(t)$ in meters travelled by a falling object after t seconds is given by

$$s(t) = 4.9t^2 \ .$$

Suppose that a ball is dropped from the roof of the Eiffel Tower, 300m above the ground.

(a) How far does the ball fall in the first 5 seconds?

(b) How long does it take the ball to reach the ground?

(c) How far does the ball travel in the interval from $t = 4$ s to $t = 5$ s?

(d) What is the average speed of the ball between $t = 4$ s and $t = 5$ s?

(e) Evaluate $\lim_{h\to 5} \frac{s(5)-s(5+h)}{h}$.

(f) What is the speed of the ball at $t = 5$ s?

(*cf.* [Stewart], Sec. 2.1, Example 3)

7. [Limits of rational functions]
Evaluate the following limits:

(a) $\displaystyle\lim_{x\to -3} \frac{x^2+5x+6}{x^2-4x-21}$

(b) $\displaystyle\lim_{h\to 0} \frac{(3+h)^2-9}{h}$

(c) $\displaystyle\lim_{x\to 3} \frac{\sqrt{x+6}-3}{x-3}$

(d) $\displaystyle\lim_{x\to\infty} \frac{2x^3-7x+4}{10x^3+2x+7}$.

(*cf.* [Stewart], Sec. 2.3, 2.5, 2.6)

8. [Limits of trigonometric functions]
   Evaluate

   (a) $\lim_{\theta \to 0} \dfrac{\sin 6\theta}{\tan 7\theta}$

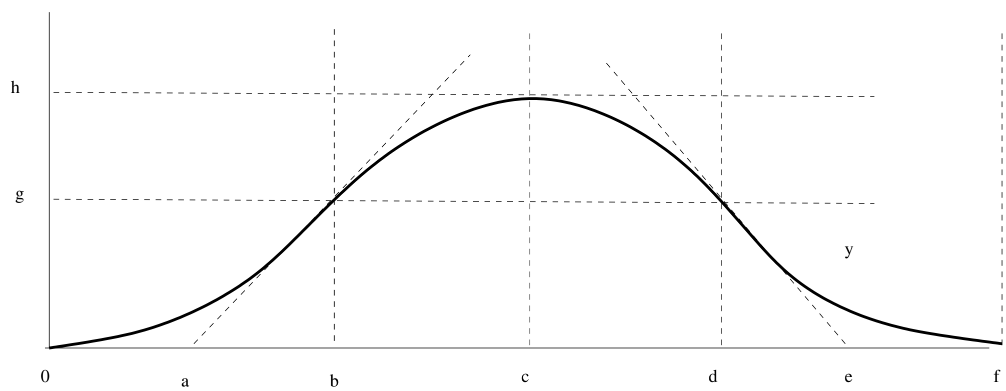
   (b) $\lim_{\theta \to 0} \dfrac{\cos\theta - 1}{\theta}$

   (c) $\lim_{x \to 0} \sin(2/x)$.

9. [Graphs]
   Sketch the graphs of the following functions.

$$f(x) = \frac{1}{x}\,, \qquad g(x) = \frac{1}{x} - 2\,, \qquad h(x) = \frac{1}{x-2}\,, \qquad k(x) = \frac{1-x}{x-2}\,.$$

10. [Graphs] A particle travels along a straight line. Its distance from a fixed point on the line at time t is a continuous function $y(t)$ whose graph is illustrated (with horizontal t-axis). There are points of inflection at $(2, 2)$ and $(4, 2)$.



   (a) On which interval(s) is the particle accelerating (i.e. $y''(t) \geq 0$)?

   (b) On which interval(s) is the particle decelerating (i.e. $y''(t) \leq 0$)?

   (c) What is the maximum speed of the particle?

   (d) At what times(s) between $t = 1$ and $t = 5$ is the particle stationary?

   (e) How far has the particle travel between $t = 0$ and $t = 3$?

10.2 Differentiation

1. [Derivative of polynomial functions]
   Let $f(x) = x^4 - 6x^2 + 4$.

   (a) Find the derivative $f'(x)$.

(b) Determine all values of x for which $f'(x) = 0$.

(c) Find the points on the curve $y = x^4 - 6x^2 + 4$ where the tangent line is horizontal.

(d) Find the equation of the tangent to the curve $y = x^4 - 6x^2 + 4$ at $x = 1$.

(*cf.* [Stewart], Sec. 3.1, Example 6.)

2. [Derivative of trigonometric functions]
Let $y = \cos t$.

   (a) Find the derivative dy/dx.

   (b) Find all values of t in the range $0 \le t \le 2\pi$ where the tangent to the curve $y = \cos t$ is horizontal.

   (c) Find the equation of the tangent to the curve $y = \cos t$ at $t = \pi/2$.

3. [Derivative of a sum]
Let $f(x) = \sin x + 2\cos x$.

   (a) Find the derivative $f'(x)$.

   (b) What is the maximum slope of a tangent to the curve $y = \sin x + 2\cos x$?

   (c) What is the minimum slope of a tangent to the curve $y = \sin x + 2\cos x$?

4. [Derivative of a product]

   (a) Find the derivative of $f(x) = (1 + 2x)\sqrt(x)$.

   (b) Find a function $f(x)$ whose derivative is $f'(x) = \sin x \cos x$.

   (*cf.* [Stewart], Sec. 3.2, Example 2.)

5. [Derivative of a quotient]
Find the derivative of the fuction $f(x) = \dfrac{x^2 + x - 2}{x^3 + 6}$.

   (*cf.* [Stewart], Sec. 3.2, Example 4.)

6. [Chain rule]
Find the derivative of $y = (x^3 - 1)^{100}$.

   (*cf.* [Stewart], Sec. 3.4, Example 3.)

7. [Chain rule]
Let $f(x) = \dfrac{1}{\sqrt[3]{x^2 + x + 1}}$. Finde $f'(x)$.

   (*cf.* [Stewart], Sec. 3.4, Example 4.)

8. [Chain rule]
Find the derivative of $y = \cos(\sin(x))$.

(*cf.* [Stewart], Sec. 3.4, Example 8.)

9. [Derivative of the exponential and logarithmic functions]

   (a) Find the derivative of $y = e^{\sin(x^2)}$.

   (b) Find the derivative of $f(x) = \ln(x^3 + 1)$.

(*cf.* [Stewart], Sec. 3.4, Example 9 and Sec. 3.6, Example 1.)

10. [Implicit differentiation]
Find y' if $y = x^3 + y^3 = 6xy$.

(*cf.* Sec. 3.5, Example 2.)

10.3 Maxima, minima and related rates

1. [maxima/minima]
Let $f(x) = x^3 - 3x^2 + 1$.

   (a) Find the critical points of $f(x)$.

   (b) For each critical point decide whether it is a local maximum or a local minimum.

   (c) Find the minimum valule of $f(x)$ on the interval $-1/2 \leq x \leq 4$.

   (d) Find the maximum value of $f(x)$ on the interval $-1/2 \leq x \leq 4$.

(*cf.* [Stewart], Sec. 4.1, Example 8)

2. [maxima/minima]
Find the absolute maximum and absolute minimum values of $f(x) = 3x^2 - 12x + 5$ on the interval $0 \leq x \leq 3$.

(*cf.* [Stewart], Sec. 4.1)

3. [maxima/minima]
A farmer wishes to fence off $900m^2$ of land adjacent to a road. It costs 40 Euro per metre to erect a fence adjacent to the road, but only 10 Euro per metre to erect a fence not adjacent to the road. Assuming the area to be fenced is rectangular, how long should the fence along the road be if the toal cost of all fencing is to be minimized?

(*cf.* [Stewart], Sec. 4.1)

4. [maxima/minima]
A box is to be made from a rectangular sheet of cardboard $70cm$ by $150cm$ by cutting

equal squares out of the four corners and bending four flaps to make the sides of the box. (The box has no top.) What is the largest possible volume of the box?

(*cf.* [Stewart], Sec. 4.1)

5. [maxima/minima]
   Find the maximum value of xy where x, y are real numbers satisfying

$$x^2 + \frac{y^2}{4} = 1.$$

   (*cf.* [Stewart], Sec. 4.1)

6. [related rates]
   A lighthouse is located on a small island 3 km away from the nearest point P on a straight shoreline and its light makes four revolutions per minute. How fast is the beam of light moving along the shoreline when it is 1 km from P ?

   (*cf.* [Stewart], Sec. 3.9)

7. [related rates]
   An aircraft is flying horizontally at a speed of 600 km/h. How fast is the distance between the aircraft and a radio beacon increasing 1 minute after the aircraft passes 5km directly above the beacon?

   (*cf.* [Stewart], Sec. 3.9)

8. [related rates]
   At a certain instant the length of a rectangle is 16m and the width is 12m. The width is increasing at 3m/s. How fast is the length changing if the area of the recangle is not changing?

   (*cf.* [Stewart], Sec. 3.9)

9. [related rates]
   A rectangular water tank is being filled at a constant rate of 20 litres per second. The base of the tank is 1 metre wide and 2 metres long. How fast is the height of the water increasing?

   (*cf.* [Stewart], Sec. 3.9)

10. [related rates]
   At noon, ship A is 100 km west of ship B. Ship A is sailing south at 35 kmh and ship B is sailing north at 25 kmh. How fast is the distance between the ships changing at 4:00 PM?

   (*cf.* [Stewart], Sec. 3.9)

10.4 Matrix algebra

1. [Linear transformations of the plane]
   Decide which of the following functions $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ are linear and, for those that are not, give an example to demonstrate non-linearity.

   (a) $f(x, y) = (x^2, y^2)$.
   (b) $f(x, y) = (2x + 3y, 4x - y)$.
   (c) $f(x, y) = (2x + 3y + 1, 4x - y)$.
   (d) $f(x, y) = (3xy, x - y)$.
   (e) $f(x, y) = (0, 0)$.
   (f) $f(x, y) = (y, x)$.

2. [Linear transformations of the plane]

   (a) Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that sends $(1, 0) \mapsto (2, 3)$ and $(0, 1) \mapsto (3, -1)$. Evaluate $f(-1, 4)$ and then find a general formula for $f(x, y)$.
   (b) Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that rotates the plane about the origin through a clockwise turn of $90°$. Evaluate $f(-1, 4)$ and then find a general formula for $f(x, y)$.
   (c) Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that reflects in the line $y = x$. Evaluate $f(-1, 4)$ and then find a general formula for $f(x, y)$.
   (d) Determine matrices that represent each of the linear functions in (a), (b) and (c).

3. [Matrix addition]
   Evaluate

   (a) $\begin{pmatrix} 1 & 2 \\ -5 & 3 \end{pmatrix} + \begin{pmatrix} -2 & 0 \\ 4 & 1 \end{pmatrix}$,
   (b) $\begin{pmatrix} 1 & 2 & 2 \\ -5 & 3 & 0 \\ 1 & 0 & -1 \end{pmatrix} + \begin{pmatrix} -2 & 0 & 7 \\ 4 & 1 & -6 \\ 0 & 0 & 5 \end{pmatrix}$,
   (c) $2\begin{pmatrix} 3 & 2 & 1 \\ -2 & 2 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 1 & 3 \\ 5 & 2 & 6 \end{pmatrix}$.
   (d) Determine the matrix representing the linear function $f(x, y) + g(x, y)$ for $f(x, y) = (x + 2y, -5x + 3y)$ and $g(x, y) = (-2x, 4x + y)$.

   (*cf.* [Lawson], Sec. 8.1)

4. [Matrix multiplication]

(a) Let $f(x, y) = (x + 2y, 3y - x)$ and $g(x, y) = (y - 2x, x + y)$. Determine a formula for the composite transformation $g \circ f \colon \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto g(f(x, y))$.

(b) Evaluate the matrix product BA where

$$B = \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}.$$

5. [Matrix multiplication]
   Let

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 4 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}, \quad D = \begin{pmatrix} 4 & -1 & 2 \end{pmatrix}.$$

   Decide which of the following arithmetic expressions can be evaluated and evaluate those that can be.

$$\text{(i)}\ DB, \quad \text{(ii)}\ BD, \quad \text{(iii)}\ AC, \quad \text{(iv)}\ CA, \quad \text{(v)}\ BD,$$

$$\text{(vi)}\ DB, \quad \text{(vii)}\ A^2, \quad \text{(viii)}\ C^2, \quad \text{(ix)}\ CAB.$$

6. [Inverse matrix]
   Let $A = \begin{pmatrix} 3 & 1 \\ 1 & -2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 5 \\ 1 & 8 \end{pmatrix}$.

   (a) Calculate A^{-1} and B^{-1}.

   (b) Verify that $(AB)^{-1} = B^{-1}A^{-1}$.

   (c) Use A^{-1} to solve the system of equations

$$\begin{aligned} 3x + y &= 13 \\ x - 2y &= 2\,. \end{aligned}$$

   (d) Find a 2×2 matrix X such that $AX = B$.

   (*cf.* [Lawson] Example 5.5.7)

7. [Inverse matrix]
   Determine a value for x for which the matrix $A = \begin{pmatrix} 5 & x \\ 2 & 4 \end{pmatrix}$ has no inverse.

8. [Inverse matrix]

(a) Consider the matrices

$$A = \begin{pmatrix} 3 & 2 & 6 \\ 4 & 3 & 4 \\ 4 & 2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & -6 & 10 \\ -4 & 15 & -12 \\ 4 & -2 & -1 \end{pmatrix}.$$

Calculate the product AB.

(b) A brewery produces Brown Ale, Dark Brown Ale and Porter. The following table summarizes the amount of malt, hops and yeast used to produce one keg of beer together with the total amount of these resources available per day.

Resource	Brown Ale	Dark Brown Ale	Porter	Daily available
Yeast	3 kg	2 kg	6 kg	250 kg
Hops	4 kg	3 kg	4 kg	220 kg
Malt	4 kg	2 kg	3 kg	170 kg

i. Let x, y, z be the number of kegs of Ale, Brown Ale and Porter produced daily. Write down a system of three linear equations which hold precisely when all three resources are fully used.

ii. Use (a) to find the values of x,y,z that ensure that the daily supply of hops, malt and yeast are fully used.

9. [Image of lines under linear transformations]
Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the mapping $f(x, y) = (x - 2y, 2x + y)$.

(a) Find the image of the line $2x + y = 4$ under f. *i.e.* Find the equation of the image.

(b) Find the pre-image of the line $2x + y = 4$ under f. *i.e.* Find the equation of the pre-image.

10. [Composite linear transformations]
Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be reflection in the line $y = x$ and let $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ be reflection in the line $y = 0$.

(a) Find the matrices of f and g.

(b) Find the matrix of the composite transformation $g \circ f\colon \mathbb{R}^2 \to \mathbb{R}^2$.

(c) Find the matrix of the composite transformation $f \circ g\colon \mathbb{R}^2 \to \mathbb{R}^2$.

10.5 Eigenvalues of 2×2 matrices

1. [Calculating determinants]
Calculate the determinants of the following matrices.

$$A = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}, B = \begin{pmatrix} 2 & 4 \\ 3 & 5 \end{pmatrix}, C = \begin{pmatrix} 4 & 5 \\ 2 & 3 \end{pmatrix}, D = \begin{pmatrix} 6 & 8 \\ 4 & 5 \end{pmatrix}.$$

(*cf.* [Lawson], Exercise 8.4.1.)

2. [Determinants & area]
Calculate the area of the parallelogram with vertices $U = (0,0), V = (2,3), W = (3,4), X = (5,7)$.

   (*cf.* [Lawson], Theorem 9.3.2)

3. [Determinants & invertibility]
Find all values of x for which the matrix $A = \begin{pmatrix} 12 & x \\ x & 18 \end{pmatrix}$ is not invertible.

4. [Eigenvalues and eigenvectors]
Find the eigenvalues and corresponding eigenvectors of the matrix $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$.

   (*cf.* [Lawson], Example 6.8.10)

5. [Eigenvalues and eigenvectors]
Let A be the 2×2 matrix representing reflection in the line $y = -x$. Find all eigenvalues and corresponding eigenvectors of A.

6. [Eigenvalues and eigenvectors]
Determine a value of x for which the matrix $A = \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix}$ has no eigenvvectors.

7. [Matrix diagonalization]
Let $A = \begin{pmatrix} -3 & 5 \\ -2 & 4 \end{pmatrix}$.

   (a) Find the eigenvalues and corresponding eignevectors of A.
   (b) Find an invertible matrix E such that $E^{-1}AE$ is diagonal.

   (*cf.* [Lawson], Example 8.6.10)

8. [Matrix diagonalization]
Let $A = \begin{pmatrix} 7 & -12 \\ 2 & -3 \end{pmatrix}$.

   (a) Find the eigenvalues and corresponding eignevectors of A.
   (b) Find an invertible matrix E such that $E^{-1}AE$ is diagonal.
   (c) Hence, or otherwise, calculate A^9.

   (*cf.* [Lawson], Example 8.6.10)

9. [Markov processes]
The Jalopy Car Rental Company has offices in Galway and Cork. Each week 90% of the cars hired out in Galway are returned to Galway and the other 10% are returned to Cork. Of the cars hired out in Cork 95% remain in Cork and 5% are returned to Galway. The

company initially has 80 cars in Galway and 60 cars in Cork. How many cars will there be in each place

(a) in one week?

(b) intwo weeks?

(c) in the long term?

10. [Markov processes]
A school of 1000 students is quarantined due to the presence of a contagious disease. Each day 20% of those that are ill become well, and 30% of those that are ill become well. Initially nobody is ill. How many students are ill after

(a) after 1 day?

(b) after 2 days?

(c) in the long run?

10.6 Number theory

1. [Clock arithmetic]
Calculate the following.

(a) $6 + 9 \ mod \ 12$

(b) $6 - 9 \ mod \ 12$

(c) $6 \times 9 \ mod \ 12$

(*cf.* [Lawson], Sec. 5.4)

2. [Clock arithmetic]
Decide which of the following inverses exist, and calculate those that do.

(a) $5^{-1} \ mod \ 9$

(b) $5^{-1} \ mod \ 10$

(c) $5^{-1} \ mod \ 11$

(d) $5^{-1} \ mod \ 12$

(e) $5^{-1} \ mod \ 15$

(f) $5^{-1} \ mod \ 16$

(*cf.* [Lawson], Sec. 5.4)

3. [ISBN]
One of the following numbers is the ISBN for *La drôle d'histoire du Finistère*. The other contains an error. Which is which?

(a) $2 - 9510 - 5011 - 2$

(b) $2 - 9150 - 5011 - 2$

4. [ISBN]
Determine the third digit of the ISBN number 3-5?0-90336-4.

5. [Euclidean algorithm]
Use the Euclidean algorithm to find integers x and y such that $gcd(a,b) = ax + by$ for each of the following pairs of numbers.

   (a) 112, 267.

   (b) 242, 1870.

   (*cf.* [Lawson], Sec. 5.2)

6. [Euclidean algorithm]
You have an unlimited supply of 3-cent stamps and an unlimited supply of 5-cent stamps. By combining stamps of different values you can make up other values: for example, three 3-cent stamps and two 5-cent stamps make the value 19 cents. What is the largest value you cannot make? Hint. You need to show that the question makes sense.

   (*cf.* [Lawson], Sec. 5.2)

7. [Euclidean algorithm & invertible numbers]

   (a) Use the Euclidean algorithm to find the inverse of 14 modulo 37.

   (b) The enciphered message

$$HVVH$$

   was produced by applying the enciphering function

$$f_E\colon \mathbb{Z}_{37} \to \mathbb{Z}_{37}, x \mapsto 14x + 20$$

   to single letter message units over the 37-letter alphabet

$$0, ..., 9, A = 10, B = 11, \ldots, Z = 35, \_ = 36 \ .$$

      i. Determine the corresponding deciphering function.

      ii. Decipher the message.

8. [Euclidean algorithm & invertible numbers]
The enciphered message

$$AEF$$

was produced by applying the enciphering function

$$f_E\colon \mathbb{Z}_{26} \to Z_{26}, x \mapsto 11x + 4$$

to single letter message units over the alphabet $A = 0, B = 1, \ldots, Z = 25$.

(i) Use the Euclidean algorithm to find the inverse of 11 modulo 26.

(ii) Determine the corresponding deciphering function.

(iii) Decipher the message.

9. [Euler Phi function]

(a) Factorise 270 as a product of primes.

(b) Calculate $\phi(270)$.

(c) Determine the number of integers from 1 to 270 that are coprime to 270.

(d) How many invertible elements are there in $\mathbb{Z}_{270}$?

10. [Euler Phi function]

(a) Factorise 1800 as a product of primes.

(b) Calculate $\phi(1800)$.

(c) Determine the number of integers from 1 to 1800 that are coprime to 1800.

(d) How many invertible elements are there in $\mathbb{Z}_{1800}$?

10.7 Integration

Chapter 5 of [Stewart] contains background and examples related to the following integration problems.

1. [Integrals as areas]
   Evaluate the following integrals.

   (i) $\int_{-2}^{3} x + 1 \, dx$

   (ii) $\int_{-2}^{3} |x + 1| \, dx$

   The *absolute value function* is defined as $|x| = x$ for $x \geq 0$, $|x| = -x$ for $x < 0$.

2. [Integrals as areas]
   Evaluate the following integrals.

   (i) $\int_{-2}^{3} \lfloor x + 1 \rfloor \, dx$

   (ii) $\int_{-2}^{3} \lfloor x + 1 \rfloor^2, dx$

   (iii) $\int_{-2}^{3} \lceil x + 1 \rceil \, dx$

   (iv) $\int_{-2}^{3} \lceil x + 1 \rceil^2, dx$

   The *floor function* is defined as $\lfloor x \rfloor = n$ where n is the largest integer $n \leq x$.

   The *ceiling function* is defined as $\lceil x \rceil = n$ where n is the smallest integer $n \geq x$.

3. [Integral of sums and scalar products]
   Evaluate the following integrals.

   (i) $\int_{-3}^{2} x + 1 + |x + 1|, dx$

   (ii) $\int_{-3}^{2} \frac{1}{4}(x + 1 + \lfloor x + 1 \rfloor), dx$

4. [Areas of bounded regions]
   Calculate the areas of the regions bounded by:

   (i) x-axis and $y = x^4 + x^2 + 1$ between $x = -1$ and $x = 1$.

   (ii) x-axis and $y = x^2 + x - 2$.

(iii) $y = x + 1$ and $y = x^2 + 2$ between $x = -1$ and $x = 1$.

(iv) $y = x^2$ and $y = 2 - x$.

5. [Fundamental Theorem of Calculus]
A particle is shot straight upwards from the ground with initial velocity $98m/s$. Its velocity after t seconds is $v = -9.8t + 98$. At what time t does it reach its maximum height? What maximum height is achieved?

6. [Fundamental Theorem of Calculus]
The rate of flow of water into an initially empty tank is $100 - 3t$ galllons per minute at time t minutes. How much water flows into the tank during the interval from $t = 10$ to $t = 20$ minutes?

7. [Fundamental Theorem of Calculus]
The birth rate in a certain city t years after 1960 was $13 + t$ thousands of births per year. Set up and evaluate an appropriate integral to compute the total number of births that occurred between 1960 and 1980.

8. [Fundamental Theorem of Calculus]
The city in the previous problem had a death rate of $5 + t/2$ thousands per year t years after 1960. If the population of the city was 125 000 in 1960, what was its population in 1980? (Consider both births and deaths.)

9. [Fundamental Theorem of Calculus]
On the moon the acceleration due to gravity is $1.6m/sec^2$. If a rock is dropped into a crevasse, how fast will it be going just before it hits the bottom $30\,sec$ later?

10. [Fundamental Theorem of Calculus]
A heavy object is dropped from the top of the Eiffel Tower. The tower is 324 metres high. Approximately how long will the object take to reach the ground? (Acceleration due to gravity is $g = 9.8m/sec^2$.)

10.8 Techniques of Integration

Chapter 7 of [Stewart] contains background and examples related to the following problems on indefinite integrals. (An indefinite integral is the same thing as an anti-derivative.)

1. [Algebraic simplification]
Determine the following integrals:

(i) $\displaystyle\int (x^2 - 1)(x + 1)\,dx$

(ii) $\displaystyle\int (x^3 + 1)^2\,dx$

2. [Simple substitution]
   Use a substitution to determine the followings integrals.

   (i) $\displaystyle\int x^3 \cos(x^4 + 2)\, dx$

   (ii) $\displaystyle\int \sqrt{2x+1}\, dx$

   (iii) $\displaystyle\int \frac{x}{\sqrt{1-4x^2}}\, dx$

3. [Logarithms]
   Determine the following integrals.

   (i) $\displaystyle\int e^{2x}\, dx$

   (ii) $\displaystyle\int \frac{1}{x}\, dx$

   (iii) $\displaystyle\int \frac{2x}{x^2+8}\, dx$

   (iii) $\displaystyle\int \frac{x^2+x}{x^3+3x+8}\, dx$

   (iv) $\displaystyle\int \frac{6x^2+4x+2}{x^3+x^2+x+8}\, dx$

4. [Integration by parts]
   Use integration by parts to determine the following integrals:

   (i) $\displaystyle\int x \sin(x)\, dx$

   (ii) $\displaystyle\int t^2 e^t\, dt$

   (iii) $\displaystyle\int e^x \sin(x)\, dx$

5. [Reduction formulae]
   Prove the reduction formula

$$\int \sin^n(x)\, dx = -\frac{1}{n}\cos(x)\sin^{n-1}(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\, dx$$

   where $n \geq 2$ is an integer.

6. [Trigonometric substitutions]
Evaluate
$$\int \frac{\sqrt{16-x^2}}{x^2}\,dx\ .$$
(Hint: Try $x = a\sin\theta$ for a suitable value of a.)

7. [Trigonometric substitutions]
Evaluate
$$\int \frac{x^3}{\sqrt{x^2+9}}\,dx\ .$$
(Hint: Try $x = a\tan\theta$ for a suitable value of a.)

8. [Partial fractions]
Evaluate
$$\int \frac{x+5}{x^2+x-2}\,dx\ .$$

9. [Partial fractions]
Evaluate
$$\int \frac{1}{x^3-x^2-x+1}\,dx\ .$$

10. [Partial fractions]
Evaluate
$$\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx\ .$$
(Hint: you'll first need to apply long division.)

10.9 Logarithms and differential equations

1. [Verifying solutions]
Show that $y = Ce^{3x} - e^{2x}$ is a solution of the differential equation
$$\frac{dy}{dx} - 3y = e^{2x}$$
for any constant C. Determine the value of C that ensures $y(0) = 3$.

2. [Verifying solutions]
Show that $y = Ae^{kt}$ is a solution of the differential equation
$$\frac{dy}{dt} = ky$$
for any constant A. Determine the values of the constants A and k that ensure $y(0) = 60$ and $y(5) = 30$.

3. [Applications of differential equations]
A cup of coffee in a room at $20°$C cools for $80°$C to $50°$C in 5 minutes. How long will it take to cool to $40°$C? (Newton's Law states that a hot object cools at a rate proportional to the excess of its temperature above room temperature.)

4. [Applications of differential equations]
The Malthusian Law states that the size $y(t)$ of a population at time t is governed by the differential equation

$$\frac{dy}{dt} = ky \ .$$

In other words, the rate of change of a population is proportional to the size of the population.

The population of the world in 1960 was 3.06 billion. Use the Malthusian Law with $k = 0.02$ to estimate the population in the year 2016.

5. [Separable differential equations]
Find the solution to the differential equation

$$y^2 \frac{dy}{dt} = t^2, \quad y(0) = 27 \ .$$

6. [Separable differential equations]
Solve the differential equation

$$e^y \frac{dy}{dt} - t - t^3 = 0, \quad y(0) = 1 \ .$$

7. [Separable differential equations]
Solve the differential equation

$$\frac{dy}{dt} = 1 + y^2, \ y(0) = 0 \ .$$

8. [Applications of differential equations]
The Logistic Law states that the size $y(t)$ of a population at time t is governed by the differential equation

$$\frac{dy}{dt} = ky = \ell y^2 \ .$$

This is a separable differential equation whose solution

$$y = \frac{Ake^{kt}}{1 + A\ell e^{kt}}$$

can be found using partial fractions.

For a given population it is estimated that $k = 0.029$ and $\ell = 2.941 \times 10^{-12}$. What will the size of this population tend to in the long term, according to the Logistic Law?

9.

10.

10.10 Logic

1. [Truth tables]
For each of the Boolean functions

   (i) $f(x, y) = x \cdot y \quad mod\ 2$

   (ii) $f(x, y) = x + y \quad mod\ 2$

   (iii) $f(x, y) = (x + y) + x \cdot y \quad mod\ 2$

   complete the following truth table.

x	y	$f(x,y)$
1	1	
1	0	
0	1	
0	0	

2. [Truth tables]
Define
$$\overline{x} = 1 + x \quad mod\ 2.$$
Write out the truth table for each of the Boolean functions

   (i) $f(x, y) = \overline{x \cdot y} \quad mod\ 2,$

   (ii) $f(x, y) = \overline{x \cdot \overline{y}} \quad mod\ 2.$

3. [Truth tables]
Write out the truth tables for the following Boolean functions. In each case decide if the function is a tautology, a contradiction or neither.

   (i) $((A \Rightarrow B) \Rightarrow B) \Rightarrow B$,

   (ii) $(A \Rightarrow B) \vee (B \Rightarrow A)$,

   (iii) $(\neg A) \Rightarrow (A \wedge B)$,

   (iv) $(A \Rightarrow B) \Leftrightarrow ((\neg A) \vee B)$.

4. [Truth tables]
   Write out the truth tables for the following Boolean functions. In each case decide if the function is a tautology, a contradiction or neither.

   (i) $(A \Leftrightarrow ((\neg B) \vee C)) \Rightarrow ((\neg A) \Rightarrow B))$,

   (ii) $(A \wedge B) \Rightarrow (A \vee C)$,

   (iii) $(A \Rightarrow (B \vee C)) \vee (A \Rightarrow B)$.

5. [Truth tables]

   (i) Decide whether or not $(\neg A) \vee B$ is logically equivalent to $(\neg B) \vee A$.

   (ii) Decide whether or not $\neg(A \Leftrightarrow B)$ is logically equivalent to $A \Leftrightarrow (\neg B)$.

6. [Logical validity]
   Represent the following argument by a truth function. Then decide if the argument is logically valid.

   > If Murphy is a Communist, Murphy is an atheist. Murphy is an atheist. Hence, Murphy is a Communist.

7. [Logical validity]
   Represent the following argument by a truth function. Then decide if the argument is logically valid.

   > If the temperature and air pressure remained constant, there was no rain. The temperature did remain constant. Therefore, if there was rain, then the air pressure did not remain constant.

8. [Logical validity]
   Represent the following argument by a truth function. Then decide if the argument is logically valid.

   > If fallout shelters are built, other countries will feel endangered and our people will get a false sense of security. If other countries feel endangered, they may start a preventative war. If our people will get a false sense of security, they will put less effort into preserving peace. If fallout shelters are not built, we run the risk of tremendous losses in the event of war. Hence, either other countries may start a preventative war and our people will put less effort into preserving peace, or we run the risk of tremendous losses in the event of war.

9. [Logical validity]
   Represent the following argument by a truth function. Then decide if the argument is logically valid.

If capital investment remains constant, then government spending will increase or unemployment will result. If government spending will not increase, taxes can be reduced. If taxes can be reduced and capital investment remains constant, then unemployment will not result. Hence government spending will increase.

10. [Logical validity]
Give a formula, in terms of the connectives $\neg$, $\wedge$ and $\vee$, for the truth function $f(x, y)$ defined by the following truth table.

x	y	$f(x,y)$
T	T	F
F	T	T
T	F	T
F	F	T

10.11 Complex numbers

1. [Basic arithmetic]
For $w = 5 + 5\mathbf{i}$ and $z = 3 - 4\mathbf{i}$ express the following complex numbers in the for $x + y\mathbf{i}$:

$$w + z, \quad w - z, \quad wz, \quad \frac{w}{z} .$$

2. [Argument and modulus]
Find the argument $Arg(z)$ and modulus $|z|$ of the following complex numbers.

(i) $z = 2 + 2\sqrt{3}\mathbf{i}$

(ii) $z = -5 + 5\mathbf{i}$

(iii) $z = \dfrac{3\mathbf{i}^{30} - \mathbf{i}^{19}}{2\mathbf{i} - 1}$

(iv) $z = \dfrac{5 + 5\mathbf{i}}{3 - 4\mathbf{i}} + \dfrac{20}{4 + 3\mathbf{i}}$

3. [De Moivre's Theorem]
Express each of the following numbers z in the form $x + y\mathbf{i}$.

(i) $z = vw$ where $v = 3(\cos 40° + \mathbf{i}\sin 40°)$ and $w = 4(\cos 80° + \mathbf{i}\sin 80°)$.

(ii) $z = v^7/w^3$ where $v = 2(\cos 15° + \mathbf{i}\sin 15°)$ and $w = 4(\cos 45° + \mathbf{i}\sin 45°)$.

(iii) $z = \left(\dfrac{1 + \sqrt{3}\mathbf{i}}{1 - \sqrt{3}\mathbf{i}}\right)^{10}$.

4. [Euler's formula]
   We define
$$e^{\mathbf{i}\theta} = \cos\theta + \mathbf{i}\sin\theta$$
   and, more generally,
$$e^{x+\mathbf{i}y} = e^x e^{\mathbf{i}y} = e^x(\cos\theta + \mathbf{i}\sin\theta)\ .$$
   Use this definition to prove the following identities.
   (i) $\cos\theta = \dfrac{e^{\mathbf{i}\theta} + e^{-\mathbf{i}\theta}}{2}$.
   (ii) $\sin\theta = \dfrac{e^{\mathbf{i}\theta} - e^{-\mathbf{i}\theta}}{2\mathbf{i}}$.
   (iii) $\cos^2\theta + \sin^2\theta = 1$.

5. [Square roots]
   Find the square roots of $-15 - 8\mathbf{i}$.

6. [Complex roots of unity]
   List all cube roots of 1. Hence factorize the polynomial $x^3 + 1$ as a product of (complex) linear polynomials.

7. [Complex roots of unity]
   Use De Moivre's Theorem to express $\cos 3\theta$ as a sum of powers of $\cos\theta$. Similarly, express $\sin 3\theta$ as a sum of powers of $\sin\theta$.

8. [Complex roots]
   Factorize $x^5 + x^4 + x^3 + x^2 + x + 1$ as a product of real linear and quadratic factors.

9. [Complex roots of unity]
   Deduce that
$$\sin\frac{\pi}{3} + \sin\frac{2\pi}{3} + \sin\frac{4\pi}{3} + \sin\frac{5\pi}{3} = 0$$
   from the fact that the nth roots of unity sum to zero.

10. [Complex roots]
    Find all complex numbers z that satisfy $z^5 = -32$. Hence factorize the polynomial
$$x^5 + 32$$
    as a product of real polynomials of degree at most 2.

10.12 Systems of Equations

1. [System of n equations in n unkowns]
   A brewery produces Brown Ale, Dark Brown Ale and Porter. The following table summarizes the amount of malt, hops and yeast used to produce one keg of beer together with the total amount of these resources available per day.

Resource	Brown Ale	Dark Brown Ale	Porter	Daily available
Yeast	3 kg	2 kg	6 kg	250 kg
Hops	4 kg	3 kg	4 kg	220 kg
Malt	4 kg	2 kg	3 kg	170 kg

   (a) Let x, y, z be the number of kegs of Ale, Brown Ale and Porter produced daily. Write down a system of three linear equations which hold precisely when all three resources are fully used.

   (b) Find the values of x,y,z which ensure that the daily supply of hops, malt and yeast are fully used.

2. [System of n equations in n unkowns]
   A small dairy produces three cheeses: mild, standard and mature. The following table summarizes the amount of energy, milk and labour used to produce one box of each of the three cheeses together with the amount of these resources available per day.

Resource	Mild A	Standard B	Mature C	Daily available
Energy	2 kWh	3 kWh	2 kWh	100 kWh
Milk	4 L	4 L	3 L	150 L
Labour	3 h	4 h	6 h	170 h

   (a) Let x, y, z be the number of boxes of mild, standard and mature cheese produced daily. Write down a system of three linear equations which hold precisely when all three resources are fully used.

   (b) Find the values of x,y,z which ensure that all resources are fully used.

3. [System of n equations in m unkowns]
   Find one solution P to the following system of linear equations.

$$\begin{array}{rcrcrcr} 3x & + & 5y & + & 7z & = & 15 \\ x & + & y & + & z & = & 1 \end{array}$$

   Then find a vector V such that all solutions are of the form $P + \lambda V$ for some scalar $\lambda \in \mathbb{R}$.

4. [System of n equations in m unkowns]
   Find one solution P to the following system of linear equations.

$$\begin{array}{rcrcrcrcr} w & + & 3x & + & 3y & + & 2z & = & 1 \\ 2w & + & 6x & + & 9y & + & 5z & = & 5 \\ -w & + & -3x & + & 3y & + & & = & 5 \end{array}$$

Then find vector V, V' such that all solutions are of the form $P + \lambda V + \mu V'$ for scalars $\lambda, \mu \in \mathbb{R}$.

5. [Inconsistent systems]
   Find all values of k for which the following system has no solutions.

$$\begin{array}{rcrcl} x & + & ky & = & 0 \\ kx & + & 9y & = & 1 \end{array}$$

6. [Inverse matices]
   Use row operations to find the inverse of

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 4 & 1 & 4 \\ 6 & 1 & 7 \end{pmatrix}.$$

7. [Inverse matices]
   Use row operations to find the inverse of

$$A = \begin{pmatrix} 3 & 2 & 6 \\ 4 & 3 & 4 \\ 4 & 2 & 3 \end{pmatrix}.$$

   (Compare Question 1.)

8. [Inverse matices]
   Use row operations to find the inverse of

$$A = \begin{pmatrix} 3 & 4 & 4 \\ 2 & 3 & 2 \\ 6 & 4 & 3 \end{pmatrix}.$$

   (Compare Question 7.)

9.

10.

11 Sample MA185 exam paper