Problem Determine

$$\int \frac{x^4 + x^2 + x}{x^2+1} \, dx$$

Sol$^n$

$$\begin{array}{r l} & x^2 \\ x^2+1 & \overline{) \, x^4 + x^2 + x} \\ & \underline{x^4 + x^2} \\ & \qquad\qquad x \end{array}$$

so $$\frac{x^4+x^2+x}{x^2+1} = x^2 + \frac{x}{x^2+1}$$

$$\int \frac{x^4+x^2+x}{x^2+1} \, dx = \int x^2 + \frac{x}{x^2+1} \, dx$$

$$= \int x^2 \, dx + \int \frac{x}{x^2+1} \, dx$$

$$= \frac{1}{3}x^3 + \frac{1}{2}\int \frac{2x}{x^2+1} \, dx$$

$$= \frac{1}{3}x^3 + \frac{1}{2} \ln(x^2+1) + C$$

Fact If a fraction is such that the top is the derivative of the bottom, then its integral is just log of the bottom.

Problem Find

$$\int \frac{\cos(x)}{\sin(x)}\, dx$$

Sol$^n$

$$\int \frac{\cos(x)}{\sin(x)}\, dx = \ln(\sin(x)) + C,$$

for $\sin(x) > 0$.

## Technique 3: simple substitution

To find $\int x \sin(x^2+3)\, dx$ we might just spot that

$$\frac{d}{dx} \cos(x^2+3) = -\sin(x^2+3) \cdot 2x$$

or

$$\frac{d}{dx} -\frac{1}{2}\cos(x^2+3) = x\sin(x^2+3)$$

and hence

$$\int x \sin(x^2+3)\, dx = -\frac{1}{2}\cos(x^2+3) + C.$$

There is an alternative method which is often easier.

To evaluate

$$I = \int x \sin(x^2+2)\, dx$$

we can substitute

$$u = x^2+2$$

$$\cancel{\frac{du}{dx} = 2x} \qquad du = 2x\, dx$$

$$\frac{1}{2} du = x\, dx$$

So

$$I = \int \frac{1}{2} \sin(u)\, du$$

$$I = -\frac{1}{2}\cos(u) + C$$

$$= -\frac{1}{2}\cos(x^2+2) + C.$$

<u>Problem</u> Evaluate

$$I = \int_{-2}^{2} (x-4)^5 \, dx$$

<u>Sol<sup></sup>n</u>

Let $u = x - 4$

$du = dx$

when $x = 2$, $u = -2$

when $x = -2$, $u = -6$

$$I = \int_{-6}^{-2} u^5 \, du$$

$$= \left. \frac{u^6}{6} \right|_{-6}^{-2}$$

$$\frac{1}{6}(-2)^6 - \frac{1}{6}(-6)^6$$

$$= \frac{1}{6}\left(+2^6 - 6^6\right)$$

$$= \text{etc.}$$

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<u>Problem</u> Evaluate

$$I = \int_1^2 x\,(x^2+5)^3\,dx$$

<u>Sol$^n$</u> Let $u = x^2+5$

$$du = 2x\,dx$$

$$\tfrac{1}{2}du = x\,dx$$

$x=2,\ u=9$

$x=1,\ u=6$

$$I = \frac{1}{2}\int_6^9 u^3\,du = \text{etc.}$$