2. Techniques of Integration

Technique 1: Algebraic simplification

Problem Determine

$$F(x) = \int \frac{x^3 + x^2 - x - 1}{x-1} \, dx .$$

Sol^n

$$\begin{array}{r|l} & x^2 + 2x + 1 \\ \hline x-1 & x^3 + x^2 - x - 1 \\ & \underline{x^3 - x^2} \\ & \quad 2x^2 - x \\ & \quad \underline{2x^2 - 2x} \\ & \qquad x - 1 \\ & \qquad \underline{x - 1} \\ & \qquad\quad 0 \end{array}$$

So $\dfrac{x^3 + x^2 - x - 1}{x-1} = x^2 + 2x + 1$

So $F(x) = \int x^2+2x+1 \; dx$

$= \frac{1}{3}x^3 + x^2 + x + C$

Problem Fixed

$F(x) = \int (x^2-1)^3 \; dx$

Soln

$(x^2-1)^3 = (x^2-1)(x^2-1)(x^2-1)$

$= x^6 - 3x^4 + 3x^2 - 1$

So $F(x) = \int x^6 - 3x^4 + 3x^2 - 1 \; dx$

$= \frac{1}{7}x^7 - \frac{3}{5}x^5 + x^3 - x + C.$

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Technique 2 : Some standard functions

1) $\int \cos(x)\,dx = \sin(x) + C$

2) $\int \sin(x)\,dx = -\cos(x) + C$

3) $\int e^x\,dx = e^x + C$

4) $\int x^n\,dx = \frac{1}{n+1} x^{n+1} + C,$
$(n \neq -1)$

5) $\int \frac{1}{x}\,dx = \,?$

To answer this let's first recall basics on logarithms.

$2^2 = 4$ means $\log_2 4 = 2$

$2^3 = 8$ means $\log_2 8 = 3$

$2^{-4} = \frac{1}{16}$ means $\log_2\left(\frac{1}{16}\right) = -4$

$3^{-3} = \frac{1}{27}$ means $\log_3\left(\frac{1}{27}\right) = -3$

$$7^{\log_7 4} = 4$$

$$2^{\log_2 8} = 8$$

$$2^{\log_2\left(\frac{1}{16}\right)} = \frac{1}{16}$$

$$\boxed{e^{\log_e x} = x}$$

$$\boxed{\log_e(e^x) = x}$$

N.B. $\log_2(-4) =$ not defined

$\log_2(0) =$ not defined.

<u>Notation</u> we write

$$\ln(x) = \log_e x \quad , \text{for } x > 0 .$$

"So"

$$\boxed{e^{\ln(x)} = x} \quad \text{for } x > 0$$

The chain rule for differentiation gives:

$$x = e^{\ln(x)}$$

$$\frac{d}{dx}(x) = \frac{d}{dx}\left(e^{\ln(x)}\right)$$

$$1 = e^{\ln(x)} \frac{d}{dx}(\ln(x))$$

$$1 = x \frac{d}{dx}(\ln(x))$$

$$\frac{1}{x} = \frac{d}{dx} \ln(x)$$

Hence

5) $\int \frac{1}{x} \, dx = \ln(x) + C$.

Problem Determine

$$\int \frac{1}{x+2}\,dx .$$

Sol^n

$$\frac{d}{dx} \ln(x+2)$$

$$= \frac{1}{x+2} \frac{d}{dx}(x+2)$$

$$= \frac{1}{x+2} .$$

Hence

$$\int \frac{1}{x+2}\,dx = \ln(x+2) + C .$$

$\square$

Problem Determine

$$\int \frac{2x+2}{x^2+2x}\,dx$$

Sol^n

$$\frac{d}{dx} \ln(x^2 + 2x)$$

$$= \frac{1}{x^2+2x} \frac{d}{dx}(x^2 + 2x)$$

$$= \frac{2x+2}{x^2+2x}$$

Hence

$$\int \frac{2x+2}{x^2+2x}\, dx = \ln(x^2+2x) + C.$$

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