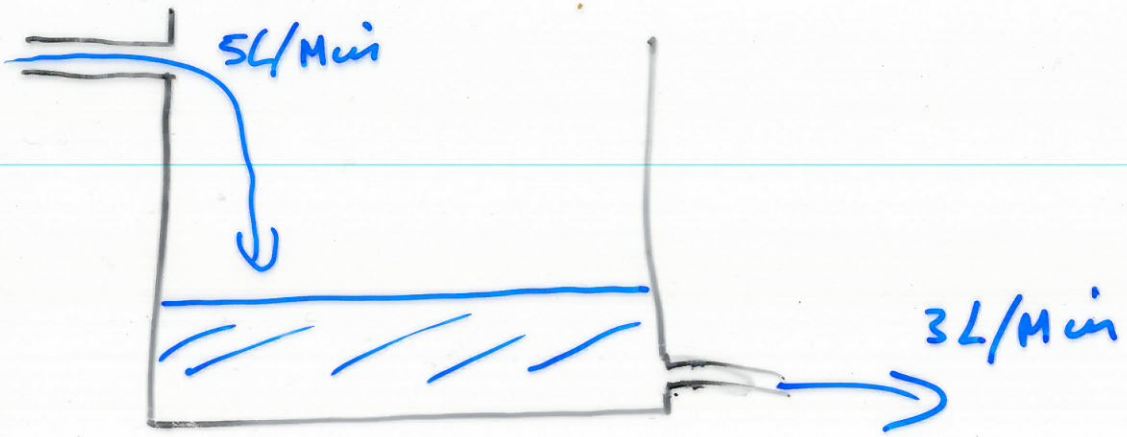
Problem A water tank contains 100L of water.



A solution with salt concentration of 0.4 Kg/L is added at a rate of 5L per minute. The solution is kept mixed and is drained from the tank at a rate of 3L per minute.

Find the concentration of salt after 20 minutes.

Sol^{n}

$y(t)$ = amount of salt in the tank (in kg) after t minutes.

$$\frac{dy}{dt} = 2 - 3\frac{y}{100+2t} \quad \text{kg/min} \qquad (*)$$

Required to find

$$\frac{y(t)}{100+2t} \text{ kg/L} \quad \text{at} \quad t = 20.$$

(*) is not separable. Let's rewrite (*) as

$$\frac{dy}{dt} + \left(\frac{3}{100+2t}\right) y = 2 \qquad (*)$$

Defn A first order linear differential equation is one of the form

$$\frac{dy}{dt} + P(t)\,y = Q(t)$$

with $P(t)$, $Q(t)$ functions of t.

So (*) is first order linear.

To solve such an equation we multiply both sides by

$$I(t) = e^{\int P(t)\,dt}$$ [integrating factor]

and integrate.

$$I(t)\frac{dy}{dt} + I(t)P(t)\,y = I(t)Q(t)$$

$$\| $$

$$\frac{d}{dt}(I(t)\,y) = I(t)Q(t)$$

integrating both sides:

$$I(t)\,y = \int I(t)\,Q(t)\,dt \quad (*)$$

Let's apply this box to

$$\frac{dy}{dt} + \underbrace{\left(\frac{3}{100+2t}\right)}_{P(t)} y = \underbrace{2}_{Q(t)}$$

$$I(t) = e^{\int \frac{3}{100+2t}\,dt}$$

$$= e^{\frac{3}{2}\int \frac{2}{100+2t}\,dt}$$

$$= e^{\frac{3}{2}\ln|100+2t|}$$

$$= e^{\ln\left(|100+2t|^{\frac{3}{2}}\right)}$$

$$= (100+2t)^{\frac{3}{2}}$$

y from (t)

$$(100+2t)^{\frac{3}{2}} y = 2\int (100+2t)^{\frac{3}{2}}\, dt$$

$$(100+2t)^{\frac{3}{2}} y = \frac{2}{2}\cdot\frac{2}{5}(100+2t)^{\frac{5}{2}}$$

$$y = \frac{2}{5}\,\frac{(100+2t)^{\frac{5}{2}}}{(100+2t)^{\frac{3}{2}}}$$

$$y = \frac{2}{5}(100+2t)$$

At $t = 20$

$$y(20) = \frac{2}{5}(140)$$

At $t=20$, the concentration of salt is

$$\frac{y(20)}{100+2(20)} = \frac{\frac{2}{5}\times 140}{140} = \frac{2}{5}\ kg/L.$$

Problem for you

Solve

$$t^2 \frac{dy}{dt} + ty = 1, \quad t > 0, \; y(1) = 2.$$

Soln Divide by t^2:

$$\frac{dy}{dt} + \frac{1}{t} y = \frac{1}{t^2}, \quad t > 0, \; y(1) = 2.$$

This is first order linear.